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On the Interaction of Mesons with the Electromagnetic Field, II

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§ 5. The correction to the current associated with the mass type term

We shall now discuss the integral (37b)

$$\delta j_\mu^b(x) = i\frac{\pi}{2}ae\int(dx')(dx'')[\psi^\dagger(x)\beta_\mu BD_F(x-x')\beta_\nu BD_F(x'-x'')\beta_\nu D_{0F}(x'-x'')\psi(x'')$$
$$+\psi^\dagger(x'')\beta_\nu BD_F(x''-x')\beta_\nu D_{0F}(x''-x')BD_F(x'-x)\beta_\mu\psi(x)]$$
$$+\frac{e}{2}\delta m\int(dx')[\psi^\dagger(x)\beta_\mu BD_F(x-x')\psi(x')+\psi^\dagger(x')BD_F(x'-x)\beta_\mu\psi(x)] \qquad (37b)$$

which involves the effect of mass renormalization on the property of the mesonic charge-current. We begin with the evaluation of

$$K(x)\equiv\beta_\nu BD_F(x)\beta_\nu D_{0F}(x). \qquad (67)$$

Substituting the Fourier integral representations (22) and (35) for $D_F(x)$ and $D_{0F}(x)$ respectively, (67) becomes

$$K(x)=-\frac{4}{(2\pi)^8}\int(dk)(dk')\,e^{i(k+k')x}\frac{\beta_\nu[i(k'\beta)+\frac{1}{m}(k'\beta)^2-m-\frac{1}{m}k'^2]\beta_\nu}{k^2(k'^2+m^2)}. \qquad (68)$$

This can also be written as

$$K(x)=-\frac{4}{(2\pi)^8}\int(dk)\,(dp)e^{ipx}\,\frac{\beta_\nu[i(p\beta-k\beta)+\frac{1}{m}(p\beta-k\beta)^2-m-\frac{1}{m}(p-k)^2]\beta_\nu}{k^2[(p-k)^2+m^2]} \qquad (69)$$

on introducing

$$p_\mu=k_\mu+k'_\mu. \qquad (70)$$

The denominator in (69) can be simplified by writing it as

$$\int_0^1 du[(k-pu)^2+m^2u^2+(p^2+m^2)u(1-u)]^{-2}. \tag{71}$$

The transformation

$$k_\mu \to k_\mu + up_\mu \tag{72}$$

brings (71) into the form

$$\int_0^1 du[k^2+m^2u^2+(p^2+m^2)u(1-u)]^{-2}. \tag{73}$$

Now this transformation (72) as well as the integration over k must be carried out carefully, because the integral (69) is divergent. Thus, after the transformation (72), $K(x)$ becomes

$$-\frac{4}{(2\pi)^8}\int (dk)\,(dp)\,e^{ipx}\int_0^1 du\left(1-up_\mu\frac{\partial}{\partial k_\mu}+\frac{1}{2}u^2p_\lambda p_\mu\frac{\partial^2}{\partial k_\lambda \partial k_\mu}\right)$$

$$\times\frac{\beta_\nu[i\{(1-u)p\beta-k\beta\}+\frac{1}{m}\{(1-u)p\beta-k\beta\}^2-m-\frac{1}{m}\{(1-u)p-k\}^2]\beta_\nu}{[k^2+m^2u^2+(p^2+m^2)u(1-u)]^2}. \tag{74}$$

Here the first and the second derivatives in the integrand represent the contributions from surface terms that must be added to take into account the effect of the displacement (72) at large values of k^2, where the integrand does not tend to zero rapidly enough. Derivatives higher than the second vanish identically, since the integral is quadratically divergent with respect to the k-integration. The integration over k in (74) can now be performed in a spherically symmetric region of the energy-momentum space.

Performing the differentiation with respect to k, the integrand in (74) becomes

$$(k^2+\Lambda^2)^{-2}\{i(1-u)(p\beta)+\frac{1}{m}(1-u)^2(p\beta)^2(1-\beta_\nu^2)-m\beta_\nu^2$$

$$+\frac{k^2}{4m}\beta_\nu^2(1-\beta_\sigma^2)\}+u\Lambda^2(k^2+\Lambda^2)^{-3}\{i(p\beta)+\frac{2}{m}(1-u)(p\beta)^2(1-\beta_\nu^2)\}$$

$$+\frac{1}{2}u^2\Big[-\frac{p^2}{m}\Lambda^2(k^2+\Lambda^2)^{-3}\beta_\nu^2(1-\beta_\sigma^2)$$

$$+\frac{1}{m}(\Lambda^2)^2(k^2+\Lambda^2)^{-4}\{2(p\beta)^2+p^2\beta_\nu^2\}(1-\beta_\sigma^2)$$

$$+2p^2\{(k^2+\Lambda^2)^{-3}-3\Lambda^2(k^2+\Lambda^2)^{-4}\}\{i(1-u)(p\beta(+\frac{1}{m}(1-u)^2(p\beta)^2(1-\beta_\nu^2)-m\beta_\nu^2\}] \tag{75}$$

where

$$\Lambda^2=m^2u^2+(p^2+m^2)u(1-u). \tag{76}$$

In deriving (75), the following substitutions

$$k_\mu \longrightarrow 0$$

$$k_\mu k_\nu \longrightarrow \frac{1}{4}k^2\delta_{\mu\nu}$$

$$k_\mu k_\nu k_\rho \longrightarrow 0$$

$$k_\mu k_\nu k_\rho k_\sigma \longrightarrow \frac{1}{24}(k^2)^2(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\rho\nu}) \tag{77}$$

have been made, in virtue of the dependence of the denominator of (74) on k^2 alone.

It is convenient to write $(k^2+\Lambda^2)^{-2}$ as

$$(k^2+m^2u^2)^{-2}+(k^2+\Lambda^2)^{-2}-(k^2+m^2u^2)^{-2}$$

$$=(k^2+m^2u^2)^{-2}-2(p^2+m^2)u(1-u)\int_0^1 dz[k^2+m^2u^2+z(p^2+m^2)u(1-u)]^{-3}. \tag{78}$$

Then one obtains

$$K(x)=-\frac{4}{(2\pi)^8}\int(dp)e^{ipx}\int_0^1 du\Big[\int(dk)\Big\{(k^2+m^2u^2)^{-2}-\int_0^1\frac{2(p^2+m^2)u(1-u)\,dz}{[k^2+m^2u^2+z(p^2+m^2)u(1-u)]^3}\Big\}$$

$$\times\Big\{i(1-u)(p\beta)+\frac{1}{m}(1-u)^2(p\beta)^2(1-\beta_\nu^2)-m\beta_\nu^2+\frac{k^2}{4m}\beta_\nu^2(1-\beta_\sigma^2)\Big\}$$

$$+\frac{i\pi^2}{2}\Big\{iu(p\beta)+\frac{1}{3m}u(6-5u)(p\beta)^2(1-\beta_\nu^2)-\frac{1}{3m}u^2p^2\beta_\nu^2(1-\beta_\sigma^2)\Big\}\Big] \tag{79}$$

using the formulas (57) and

$$\int(dk)\frac{1}{(k^2+\Lambda^2)^4}=\frac{i\pi^2}{6(\Lambda^2)^2}. \tag{80}$$

Now, the integrand in the square bracket can be rearranged in the form of ascending powers with respect to $ip\beta+m$ as follows

$$\Big[\int(dk)(k^2+m^2u^2)^{-2}\Big\{-m(1-u)(2-u)-m\beta_\nu^2u(2-u)+\frac{k^2}{4m}\beta_\nu^2(1-\beta_\sigma^2)\Big\}$$

$$+\frac{i\pi^2}{2}\Big\{-mu-\frac{1}{3}mu(6-5u)(1-\beta_\nu^2)+\frac{1}{3}mu^2\beta_\nu^2(1-\beta_\sigma^2)\Big\}\Big]$$

$$+(ip\beta+m)\Big[\int(dk)(k^2+m^2u^2)^{-2}\{(1-u)(3-2u)-2\beta_\nu^2(1-u)^2\}$$

$$+\frac{i\pi^2}{2}\Big\{u+\frac{2}{3}u(6-5u)(1-\beta_\nu^2)\Big\}\Big]$$

$$+(p^2+m^2)\Big[-2u(1-u)\int(dk)(k^2+m^2u^2)^{-3}\Big\{-m(1-u)(2-u)-m\beta_\nu^2u(2-u)$$

$$+\frac{k^2}{4m}\beta_\nu^2(1-\beta_\sigma^2)\Big\}-\frac{i\pi^2}{6m}u^2\beta_\nu^2(1-\beta_\sigma^2)\Big]+0((ip\beta+m)^2) \tag{81}$$

where $ip\beta+m$ may be laid on the right-hand side of the square brackets. The integration with respect to z is performed in a trivial way for the lowest power term of (p^2+m^2)-expansion. Therefore (79) becomes

$$K(x)=-\frac{4}{(2\pi)^4}\delta(x)\int_0^1 du\Big[\int(dk)(k^2+m^2u^2)^{-2}\Big\{-m(1-u)(2-u)-m\beta_\nu^2u(2-u)$$
$$+\frac{k^2}{4m}\beta_\nu^2(1-\beta_o^2)\Big\}+\frac{i\pi^2}{2}\Big\{-mu-\frac{1}{3}mu(6-5u)(1-\beta_\nu^2)+\frac{1}{3}mu^2\beta_\nu^2(1-\beta_o^2)\Big\}\Big]$$
$$-\frac{4}{(2\pi)^4}(\beta\partial+m)\delta(x)\int_0^1 du\Big[\int(dk)(k^2+m^2u^2)^{-2}\{(1-u)(3-2u)-2\beta_\nu^2(1-u)^2\}$$
$$+\frac{i\pi^2}{2}\Big\{u+\frac{2}{3}u(6-5u)(1-\beta_\nu^2)\Big\}\Big]$$
$$-\frac{4}{(2\pi)^4}(-\Box^2+m^2)\,\delta(x)\int_0^1 du\Big[-\frac{1}{2m}u(1-u)\beta_\nu^2(1-\beta_o^2)\int(dk)(k^2+m^2u^2)^{-2}$$
$$+\frac{i\pi^2}{mu}\Big\{(1-u)^2(2-u)+\beta_\nu^2u(1-u)(2-u)+\frac{1}{4}u^2\Big(1-\frac{5}{3}u\Big)\beta_\nu^2(1-\beta_o^2)\Big\}\Big] \qquad (82)$$
$$+\cdots\cdots\cdots\cdots$$

The first term in (82) is one which should be cancelled by the counter self-energy term in (37b). In order to show that this is really the case, we shall begin with the consideration of self-energy. The electromagnetic self-energy of a free meson can be separated easily from the S-matrix, which is in the e^2-approximation

$$H^{\rm self}(x)=\frac{1}{2}[\psi^\dagger(x)\phi(x)+\phi^\dagger(x)\psi(x)] \qquad (83)$$

with

$$\phi(x)=-i\pi u\int(dx')\beta_\mu BD_F(x-x')\beta_\mu D_{0F}(x-x')\psi(x'). \qquad (84)$$

This is treated almost in the same way as above with the only difference that we can now identify $ip\beta+m$ in (82) with zero on account of the fact that $\psi(x)$ satisfies the Duffin-Kemmer equation for the free meson. We can thus find that

$$H^{\rm self}(x)=\frac{4i\pi u}{(2\pi)^4}\psi^\dagger(x)\int_0^1 du\Big[\int(dk)(k^2+m^2u^2)^{-2}\Big\{-m(1-u)(2-u)-m\beta_\nu^2u(2-u)$$
$$+\frac{k^2}{4m}(\eta-4\beta_\nu^2)\Big\}+\frac{i\pi^2}{2}\Big\{-mu-\frac{1}{3}mu(6-5u)(1-\beta_\nu^2)+\frac{1}{3}mu^2(\eta-4\beta_\nu^2)\Big\}\Big]\psi(x) \qquad (85)$$

where

$$\eta\equiv(5-\beta_\nu^2)\beta_o^2=\begin{cases}6\times\mathbf{1}_{10} & \text{for vector mesons}\\ 4\times\mathbf{1}_5 & \text{for scalar mesons}\end{cases} \qquad (86)$$

and $\mathbf{1}_k$ is a unit matrix in a k-dimensional space. Remembering the remark concerning the formula (26), we see that (85) gives rise to the same effect in the S-matrix as

$$\delta m\psi^\dagger(x)\psi(x) \tag{87}$$

with

$$\delta m=\frac{2ia}{(2\pi)^3}\int_0^1 du\Big[\int(dk)(k^2+m^2u^2)^{-2}\Big\{-m(1-u)(2-u)-\frac{5}{2}mu(2-u)+\frac{k^2}{4m}(\eta-10)\Big\}$$
$$+\frac{i\pi^2}{2}m\Big\{2u-\frac{5}{2}u^2+\frac{1}{3}u^2(\eta-10)\Big\}\Big] \tag{88}$$

which is obtained by replacing β_o^2 in (85) with $5/2$. What we want to show is that (87) can be used as the counter self-energy term of our theory inhte e^2-approximation. We shall write, for convenience, the self-energy (85) as

$$H^{\text{self}}(x)=A\psi^\dagger(x)\psi(x)+A'\psi^\dagger(x)\beta_o^2\psi(x), \tag{89}$$

then

$$\delta m=A+\frac{5}{2}A'. \tag{90}$$

The first term in (82), i.e.,

$$\frac{i}{\pi a}(A+A'\beta_o^2)\delta(x) \tag{91}$$

together with the counter self-energy term then gives to the current $\delta j_\mu^{\flat}(x)$ the following contributions

$$i\frac{\pi}{2}ae\int(dx')(dx'')\Big[\psi^\dagger(x)\beta_\mu BD_F(x-x')\frac{i}{\pi a}(A+A'\beta_o^2)\delta(x'-x'')\psi(x'')$$
$$+\psi^\dagger(x'')\frac{i}{\pi a}(A+A'\beta_\rho^2)\delta(x'-x'')BD_F(x'-x)\beta_\mu\psi(x)\Big]$$
$$+\frac{e}{2}\delta m\int(dx')[\psi^\dagger(x)\beta_\mu BD_F(x-x')\psi(x')+\psi^\dagger(x')BD_F(x'-x)\beta_\mu\psi(x)]$$
$$=\frac{e}{2}A'\int(dx')\Big[\psi^\dagger(x)\beta_\mu BD_F(x-x')\Big(\frac{5}{2}-\beta_\rho^2\Big)\psi(x')+\psi^\dagger(x')\Big(\frac{5}{2}-\beta_\rho^2\Big)BD_F(x'-x)\beta_\mu\psi(x)\Big]. \tag{92}$$

Now, since it is proved that

$$\psi^\dagger(x')\Big(\frac{5}{2}-\beta_\rho^2\Big)(mBD_F(x'-x)-i\delta(x'-x))=\frac{1}{4}\partial_\nu'[\psi^\dagger(x')\{\beta_\nu\beta_\rho^2-\beta_\rho^2\beta_\nu\}BD_F(x'-x)] \tag{93}$$

employing (1′) and

$$(\beta\partial+m)BD_F(x)=2i\delta(x), \tag{94}$$

(92) becomes

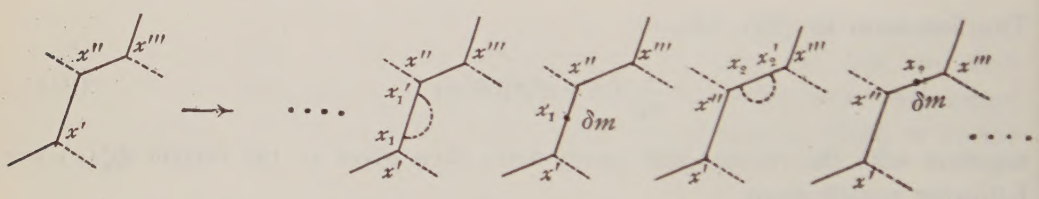
$$\frac{ie}{2m}A'\int(dx')\left[\psi^\dagger(x)\beta_\mu\delta(x-x')\left(\frac{5}{2}-\beta_\rho^2\right)\psi(x')+\psi^\dagger(x')\left(\frac{5}{2}-\beta_\rho^2\right)\delta(x'-x)\beta_\mu\psi(x)\right]$$

$$=\frac{ie}{2m}A'\left[\psi^\dagger(x)\beta_\mu\left(\frac{5}{2}-\beta_\rho^2\right)\psi(x)+\psi^\dagger(x)\left(\frac{5}{2}-\beta_\rho^2\right)\beta_\mu\psi(x)\right], \tag{95}$$

which is equal to zero on account of

$$\beta_\mu\beta_\rho^2+\beta_\rho^2\beta_\mu=5\beta_\mu. \tag{96}$$

This proof can easily be extended to include general cases. To see that it is really so, we shall observe a single meson line which consists of several internal meson lines and zero or two external meson lines and which has no vertex without a photon line. We now insert a graph of e^2-self-energy as well as that of the counter-self-energy term into this diagram in all possible ways. Then we obtain a series of modified graphs whose representative parts are described in Fig. 5. When the self-energy term is expanded into the ascending powers of $\beta\partial+m$, the first term is easily found to involve the same factor $A+A'\beta_\rho^2$ as $K(x)$ (see (82) and (91)).



(original graph) (modified graphs)

Fig. 5.

Therefore, we have only to show that we can generally determine the value of δm as $A+\frac{5}{2}A'$.

By analogous discussion to the one made in deriving (92), the modified graphs in Fig. 5 are found to give rise to terms

$$\cdots\cdots\cdots\cdots$$

$$+\cdots\cdots BD_F(x'-x_1)\left(\frac{5}{2}-\beta_\rho^2\right)BD_F(x_1-x'')\beta_\mu BD_F(x''-x''')\cdots\cdots$$

$$+\cdots\cdots BD_F(x'-x'')\beta_\mu BD_F(x''-x_2)\left(\frac{5}{2}-\beta_\rho^2\right)BD_F(x_2-x''')\cdots\cdots \tag{97}$$

$$+\cdots\cdots\cdots\cdots$$

in the integrand of the matrix element in consideration, where only the terms involving the vertex x'' are given explicitly and δm is equated to $A+\frac{5}{2}A'$.

The integral of (97) is equal to that of

$$
\begin{aligned}
&\cdots\cdots\cdots\cdots\\
&+\cdots\Big[\frac{i}{m}\Big(\frac{5}{2}-\beta_\rho^2\Big)BD_F(x'-x'')\beta_\mu BD_F(x''-x''')\\
&\qquad\qquad+\frac{i}{m}BD_F(x'-x'')\Big(\frac{5}{2}-\beta_\rho^2\Big)\beta_\mu BD_F(x''-x''')\Big]\cdots\\
&+\cdots\Big[\frac{i}{m}BD_F(x'-x'')\beta_\mu\Big(\frac{5}{2}-\beta_\rho^2\Big)BD_F(x''-x''')\\
&\qquad\qquad+\frac{i}{m}BD_F(x'-x'')\beta_\mu BD_F(x''-x''')\Big(\frac{5}{2}-\beta_\rho^2\Big)\Big]\cdots \qquad (98)\\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

using the identity

$$
\begin{aligned}
&BD_F(x'-x)\Big(\frac{5}{2}-\beta_\rho^2\Big)BD_F(x-x'')-\frac{i}{m}\delta(x'-x)\Big(\frac{5}{2}-\beta_\rho^2\Big)BD_F(x-x'')\\
&\qquad\qquad-\frac{i}{m}BD_F(x'-x)\Big(\frac{5}{2}-\beta_\rho^2\Big)\delta(x-x'')\\
&=\frac{1}{4m}\partial_\nu[BD_F(x'-x)\{\beta_\rho^2\beta_\nu-\beta_\nu\beta_\rho^2\}BD_F(x-x'')] \qquad (99)
\end{aligned}
$$

(98) is easily found to vanish identically on employing (96) when our meson line involves no external line. When it is not the case, we must apply (93) in place of (99) to the vertex which is connected to the external meson line. The above conclusion is, of course, unchanged. This is the result which we have intended to prove.

To make our statements complete, we shall give here the explicit form for the electromagnetic mass of the free mesons. By a partial integration of (88) with respect to u, one obtains

$$
\begin{aligned}
\delta m=\frac{2ia}{(2\pi)^3}\Big[&\frac{1}{4m}(\eta-10)\int(dk)\,(k^2+m^2)^{-1}+\frac{m}{10}(\eta-40)\int(dk)\,(k^2+m^2)^{-2}\\
&-2i\pi^2m\int_0^1du\Big(2+u+\frac{1}{3}u^2-\frac{\eta}{12}u^2\Big)\\
&+\frac{i\pi^2}{2}m\int_0^1du\Big(2u-\frac{5}{2}u^2+\frac{1}{3}u^2(\eta-10)\Big)\Big] \qquad (100)
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
-\frac{a}{2\pi}\Big[&\frac{1}{4m}(\eta-10)\Big(KK_0-m^2\log\frac{K+K_0}{m}\Big)\\
&+\frac{m}{12}(\eta-40)\Big(\log\frac{K+K_0}{m}-1\Big)
\end{aligned}
$$

$$-\frac{m}{36}(94-\eta)-\frac{m}{36}\left(\frac{17}{2}-\eta\right)\Big]$$

$$=\frac{\alpha}{2\pi}m\Big[\frac{1}{4}(10-\eta)\frac{KK_0}{m^2}+\frac{1}{6}(5+\eta)\log\frac{K+K_0}{m}+\frac{1}{24}(2\eta-23)\Big] \tag{101}$$

using the formulas (55) and (56). When expressed in respective representations, the electromagnetic masses are therefore

$$\delta m=\frac{3\alpha}{4\pi}m\Big[\frac{KK_0}{m^2}+\log\frac{K+K_0}{m}-\frac{5}{12}\Big] \tag{102}$$

for scalar mesons and

$$\delta m=\frac{\alpha}{2\pi}m\Big[\frac{KK_0}{m^2}+\frac{11}{6}\log\frac{K+K_0}{m}-\frac{11}{24}\Big] \tag{103}$$

for vector mesons.

Since the term (92) has thus disappeared, only the remaining parts of (82) give contributions to the current

$$\begin{aligned}\delta j_\mu^3(x)=&-\frac{i\alpha e}{(2\pi)^3}\int(dx')(dx'')\{\psi^\dagger(x)\beta_\mu BD_F(x-x')(\beta\partial'+m)\delta(x'-x'')P(\beta)\psi(x'')\\&+\psi^\dagger(x'')P(\beta)(\beta\partial''+m)\delta(x''-x')BD_F(x'-x)\beta_\mu\psi(x)\}\\&-\frac{i\alpha e}{(2\pi)^3}\int(dx')(dx'')\{\psi^\dagger(x)\beta_\mu BD_F(x-x')(-\Box'^2+m^2)\delta(x'-x'')Q(\beta)\psi(x'')\\&+\psi^\dagger(x'')Q(\beta)(-\Box''^2+m^2)\delta(x''-x')BD_F(x'-x)\beta_\mu\psi(x)\}\\&+\cdots\cdots\cdots\cdots\end{aligned} \tag{104}$$

where we have employed the following abbreviations

$$\begin{aligned}P(\beta)\equiv P_1+P_2\beta_\rho^2\equiv\int_0^1 du\Big[&\{(1-u)(3-2u)-2\beta_\rho^2(1-u)^2\}\int(dk)(k^2+m^2u^2)^{-2}\\&+\frac{i}{2}\pi^2\Big\{u+\frac{2}{3}u(6-5u)(1-\beta_\rho^2)\Big\}\Big],\end{aligned} \tag{105}$$

$$\begin{aligned}Q(\beta)\equiv Q_1+Q_2\beta_\rho^2\equiv\int_0^1 du\Big[&-\frac{1}{2m}u(1-u)(\eta-4\beta_\rho^2)\int(dk)(k^2+m^2u^2)^{-2}\\&+\frac{i\pi^2}{mu}\Big\{(1-u)^2(2-u)+\beta_\rho^2u(1-u)(2-u)+\frac{1}{4}u^2\Big(1-\frac{5}{3}u\Big)(\eta-4\beta_\rho^2)\Big\}\Big].\end{aligned} \tag{106}$$

After the partial integration with respect to x' the integration of (104) can now easily be carried out with the result

$$\frac{2\alpha e}{(2\pi)^3}\psi^\dagger(x)\{\beta_\mu P(\beta)+P(\beta)\beta_\mu-\beta_\mu B(x)Q(\beta)-Q(\beta)B^\dagger(-x)\beta_\mu\}\psi(x) \tag{107}$$

$$+\cdots\cdots\cdots\cdots$$

where the formulas (24) and (94) have been employed. $B^{\dagger}(-x)$ means that the differentiation involved in it operates not on $\psi(x)$ but on $\psi^{\dagger}(x)$. (107) can be simplified as

$$\frac{2ue}{(2\pi)^3}\psi^{\dagger}(x)\beta_{\mu}\psi(x)\,(2P_1+5P_2+4mQ_1+10mQ_2) \tag{108}$$

on making use of the Duffin-Kemmer equations (1) and (1′) satisfied by the free meson field. In (108) only the first term of (107) survives. The divergent constants P's and Q's are evaluated in a familiar way, the results being

$$\begin{aligned}
P_1&=i\pi^2\Big(\frac{7}{3}\log\frac{K+K_0}{m}+\frac{83}{36}\Big),\\
P_2&=i\pi^2\Big(-\frac{4}{3}\log\frac{K+K_0}{m}-\frac{14}{9}\Big),\\
Q_1&=\frac{i\pi^2}{m}\Big(-\frac{\eta}{6}\log\frac{K+K_0}{m}+2\log\frac{m}{2k_{\min}}-\frac{4}{3}+\frac{\eta}{72}\Big),\\
Q_2&=\frac{i\pi^2}{m}\Big(\frac{2}{3}\log\frac{K+K_0}{m}+\frac{7}{9}\Big)
\end{aligned} \tag{109}$$

where $k_{\min}$ is an invariant minimum light quantum wave number and therefore the term $\log(m/2k_{\min})$ indicates the divergence associated with zero frequency light quanta, i.e., the " infra-red catastrophe ". Inserting these values into (108), one obtains the following result

$$\frac{u}{4\pi}\Big\{\frac{2}{3}(7-\eta)\log\frac{K+K_0}{m}+8\log\frac{m}{2k_{\min}}+\frac{1}{18}(\eta-13)\Big\}j_{\mu}(x). \tag{110}$$

It is to be noted, however, that (110) is not the only result that can be expected from (104). In fact, there are many other ways of computing it. For example, we may integrate (104) after the partial integration with regard to x'' (not to x') obtaining a vanishing value for it.

The correct procedure to treat (104) has been prescribed by Dyson[10] and explained in detail by Karplus and Kroll[14] in the case of electrons which can of course be brought unchanged into our case. The above way of computing (104) is the same with evaluating the modified function $BD'_F(x)$ corresponding to the propagation function of a virtual meson $BD_F(x)$. Among the correction terms of the latter is a (divergent) term which is proportional to the original $BD_F(x)$ and therefore $BD'_F(x)$ is expressed as

$$BD'_F(x)=Z_2BD_F(x)+\cdots\cdots \tag{111}$$

with

$$Z_2=1+uR+\cdots\cdots, \tag{112}$$

where R is in a close relation to the coefficient of the current in (110). Now in

order to preserve the unitarity of the matrix S, it is necessary to divide by Z_2^n the matrix element of S between states containing n mesons, as was shown by the above authors. This is equivalent to putting equal to $Z_2^{1/2}=Z_2/Z_2^{1/2}$ the renormalization factors in eqs. (113) ;

$$\psi'(x)=Z_2^{1/2}\psi(x)$$

and

$$\psi^{\dagger\prime}(x)=Z_2^{1/2}\psi^\dagger(x), \tag{113}$$

which represent the modification of the meson wave functions brought about by virtual interaction with the electromagnetic field. (113) is exact since $\psi(x)$ and $\psi^\dagger(x)$ obey the Duffin-Kemmer equations for free mesons. (A straightforward calculation of $\psi'(x)$ yields of course a result with the coefficient Z_2.) It is now evident that the correct expression for $\delta j_\mu^b(x)$ is obtained by multiplying (110) with $1/2$ since the second term of $Z_2^{1/2}=1+\frac{1}{2}aR+\cdots$ involves a factor $1/2$. Therefore, one finally obtains

$$\delta j_\mu^b(x)=\frac{a}{8\pi}\left\{\frac{2}{3}(7-\eta)\log\frac{K+K_0}{m}+8\log\frac{m}{2k_{\min}}+\frac{1}{18}(\eta-13)\right\}j_\mu(x). \tag{114}$$

§ 6. The correction of Lamb shift type

We shall finally be concerned with the third correction (37c)

$$\delta j_\mu^c(x)=i\frac{\pi}{2}ae\int(dx')(dx'')\psi^\dagger(x')\beta_\nu BD_F(x'-x)\beta_\mu BD_F(x-x'')\beta_\nu\psi(x'')D_{0F}(x'-x'') \tag{37c}$$

or

$$\delta j_\mu^c(x)=i\frac{\pi}{2}ae\int(dx')(dx'')\psi^\dagger(x')K_\mu(x'-x,x-x'')\psi(x'') \tag{115}$$

with

$$K_\mu(x'-x,x-x'')=\beta_\nu BD_F(x'-x)\beta_\mu BD_F(x-x'')\beta_\nu D_{0F}(x'-x'') \tag{116}$$

which can be evaluated inserting the Fourier integral representations (22) and (35) for the D-functions involved. On performing this substitution, (116) becomes

$$K_\mu(x'-x,x-x'')=\frac{(-2i)^3}{(2\pi)^{12}}\int(dk)(dk')(dk'')e^{i(k+k')(x'-x)}e^{i(k+k'')(x-x'')}$$
$$[(k'^2+m^2)(k''^2+m^2)k^2]^{-1}$$
$$\times\beta_\nu\left\{i(k'\beta)+\frac{1}{m}(k'\beta)^2-m-\frac{1}{m}k'^2\right\}\beta_\mu\left\{i(k''\beta)+\frac{1}{m}(k''\beta)^2-m-\frac{1}{m}k''^2\right\}\beta_\nu. \tag{117}$$

It is convenient to replace k_μ' and k_μ'' by

$$p_\mu'=k_\mu+k_\mu',\quad p_\mu''=k_\mu+k_\mu'' \tag{118}$$

which is effectively subject to the restriction

$$p'^2+m^2=p''^2+m^2=0, \tag{119}$$

since $K_\mu(x'-x, x-x'')$ is to be multiplied by $\psi^\dagger(x')$ and $\psi(x'')$ and integrated with respect to x' and x''. As a result of this transformation, (117) becomes

$$\begin{aligned}K_\mu(x'-x, x-x'')=&\frac{(-2i)^3}{(2\pi)^{12}}\int(dk)\,(dp')\,(dp'')\,e^{ip'(k'-x)}e^{ip''(x-x'')}\\&\times[\{(p'-k)^2+m^2\}\{p''-k)^2+m^2\}k^2]^{-1}\\&\times\beta_\nu\left\{i(p'\beta-k\beta)+\frac{1}{m}(p'\beta-k\beta)^2-m-\frac{1}{m}(p'-k)^2\right\}\beta_\mu\\&\times\left\{i(p''\beta-k\beta)+\frac{1}{m}(p''\beta-k\beta)^2-m-\frac{1}{m}(p''-k)^2\right\}\beta_\nu,\end{aligned} \tag{120}$$

which can be rewritten as

$$\begin{aligned}&\frac{(-2i)^3}{(2\pi)^{12}}\int(dk)\,(dp')\,(dp'')\,e^{ip'(x'-x)}\,e^{ip''(x-x'')}\int_{-1}^{+1}dv\int_0^1 udu[k^2-k\{p'+p''+v(p'-p'')\}u]^{-3}\\&\times\beta_\nu\left\{i(p'\beta-k\beta)+\frac{1}{m}(p'\beta-k\beta)^2-m-\frac{1}{m}(p'-k)^2\right\}\beta_\mu\\&\times\left\{i(p''\beta-k\beta)+\frac{1}{m}(p''\beta-k\beta)^2-m-\frac{1}{m}(p''-k)^2\right\}\beta_\nu\end{aligned} \tag{121}$$

since

$$[\{(p'-k)^2+m^2\}\{(p''-k)^2+m^2\}k^2]^{-1}=\int_{-1}^{+1}dv\int_0^1 udu[k^2-k\{p'+p''+v(p'-p'')\}u]^{-3}. \tag{122}$$

$K_\mu(x'-x, x-x'')$ diverges logarithmically on integration with respect to k, and therefore it must be treated carefully. Thus the transformation

$$k_\mu\to k_\mu+\frac{u}{2}\{p_\mu'+p_\mu''+v(p_\mu'-p_\mu'')\} \tag{123}$$

brings (121) into

$$\begin{aligned}&\frac{(-2i)^3}{(2\pi)^{12}}\int(dk)\,(dp')\,(dp'')\int_{-1}^{+1}dv\int_0^1 udu\,e^{ip'(x'-x)}\,e^{ip''(x-x'')}\\&\times\left[1-\frac{u}{2}\{p_\rho'+p_\rho''+v(p_\rho'-p_\rho'')\}\frac{\partial}{\partial k_\rho}\right]\left[\mathfrak{N}_\mu\left\{k^2+m^2u^2+\frac{1}{4}(p'-p'')^2u^2(1-v^2)\right\}^{-3}\right]\end{aligned} \tag{124}$$

with

$$\mathfrak{K}_\mu=\beta_\nu\Big[i\Big\{\Big(1-\frac{u}{2}(1+v)\Big)(p'\beta)-\frac{u}{2}(1-v)(p''\beta)\Big\}-i(k\beta)-m$$
$$+\frac{1}{m}\Big(1-\frac{u}{2}(1+v)\Big)(p'\beta)-\frac{u}{2}(1-v)(p''\beta)\Big\}^2$$
$$-\frac{1}{m}\Big\{\Big(1-\frac{u}{2}(1+v)\Big)p'-\frac{u}{2}(1-v)p''\Big\}^2\Big]$$
$$\times\beta_\mu\Big[i\Big\{-\frac{u}{2}(1+v)\Big)(p'\beta)+\Big(1-\frac{u}{2}(1-v)\Big)(p''\beta)\Big\}-i(k\beta)-m$$
$$+\frac{1}{m}\Big\{-\frac{u}{2}(1+v)(p'\beta)+\Big(1-\frac{u}{2}(1-v)\Big)(p''\beta)\Big\}^2$$
$$-\frac{1}{m}\Big\{-\frac{u}{2}(1+v)p'+\Big(1-\frac{u}{2}(1-v)\Big)p''\Big\}^2\Big]\beta_\nu \qquad (125)$$

where the derivative with respect to k in the integrand represents the surface integral contribution which is necessary to identify (124) with (121). Higher derivatives are unnecessary, since $K_\mu(x'-x, x-x'')$ diverges only logarithmically. To evaluate this integral we must first rearrange the integrand into the form in which $ip'\beta+m$ is placed on the left-hand side while $ip''\beta+m$ on the right. Now, a left-hand factor $ip'\beta+m$ is equivalent to $\beta\partial'+m$ operating on $K_\mu(x'-x, x-x'')$, which annihilates $\psi^\dagger(x')$ on integration by parts. Similarly, a right-hand factor $ip''\beta+m$ annihilates $\psi(x'')$. In performing this rearrangement, the following formulas

$$\partial_\mu''\psi(x'')=\beta_\mu(\beta\partial'')\psi(x''), \qquad \partial_\mu'\psi^\dagger(x')=\partial_\nu'\psi^\dagger(x')\beta_\nu\beta_\mu,$$
$$(\partial_\mu''+s_{\mu\nu}\partial_\nu''+m\beta_\mu)\psi(x'')=0, \qquad \partial_\mu'\psi^\dagger(x')+\partial_\nu'\psi^\dagger(x')s_{\nu\mu}-m\psi^\dagger(x')\beta_\mu=0 \qquad (126)$$

or their Fourier transforms

$$p_\mu''=p_\nu''\beta_\mu\beta_\nu, \qquad p_\mu'=p_\nu'\beta_\nu\beta_\mu,$$
$$p_\mu''+s_{\mu\nu}p_\nu''-im\beta_\mu=0, \qquad p_\mu'-s_{\mu\nu}p_\nu'-im\beta_\mu=0 \qquad (127)$$

are useful which can easily be obtained from eqs. (1) and (1'). $s_{\mu\nu}$ is given by

$$s_{\mu\nu}=\beta_\mu\beta_\nu-\beta_\nu\beta_\mu. \qquad (128)$$

After a straightforward but tedious calculation, (124) becomes

$$\frac{(-2i)^3}{(2\pi)^{12}}\int(dk)(dp')(dp'')\int_{-1}^{+1}dv\int_0^1 u\,du\,\exp\Big[i\frac{p'+p''}{2}(x'-x'')\Big]\exp\Big[i(p''-p')\Big(x-\frac{x'+x''}{2}\Big)\Big]$$
$$\times\Big[k^2+m^2u^2\Big\{1+\frac{1}{4m^2}(p'+p'')^2(1-v^2)\Big\}\Big]^{-3}$$

$$\times\Big[\frac{k^2}{4}\beta_\mu\Big\{10-8u-2\eta(1-n)+\frac{(p'-p'')^2}{m^2}\Big(-1+\frac{7}{4}u\Big)\Big\}$$

$$+\frac{k^2}{4}is_{\mu\nu}\frac{(p_\nu'-p_\nu'')}{m}\Big\{2+\frac{u}{2}(9-2\eta)+\beta_\rho{}^2(-4+3u)\Big\}$$

$$+\beta_\mu\Big\{m^2(2-u)^2+(p'-p'')^2(2-u)\Big(1-u+\frac{u^2}{2}\Big)+\frac{(p'-p'')^2}{8}u^2(1-3u)(1-v^2)$$

$$+\frac{((p'-p'')^2)^2}{16m^2}u^2(1-u)(1-v^2)\Big\}$$

$$+ims_{\mu\nu}(p_\nu'-p_\nu'')\Big\{-\frac{u}{2}(2-u)^2+\frac{\beta_\rho{}^2}{2}u^2(2-u)+u^2(1-u)(1-v^2)\Big(\frac{5}{4}-\frac{1}{2}\beta_\rho{}^2\Big)$$

$$+\frac{(p'-p'')^2}{8m^2}u^2(\beta_\rho{}^2-u)(1-v^2)\Big\}$$

$$-\frac{u^2}{6}\Big\{1+\frac{(p'-p'')^2}{4m^2}(1-v^2)\Big\}\Big\{m^2\beta_\mu u\Big(2\eta-8+\frac{7}{4m^2}(p'-p'')^2\Big)$$

$$+ims_{\mu\nu}(p_\nu'-p_\nu'')u\Big(\frac{9}{2}-\eta+3\beta_\rho{}^2\Big)\Big\}\Big]. \tag{129}$$

In writing (129) we have exploited the symmetry of the denominator in (124) in connection with the k integration, and discarded terms involving k_ρ's in odd powers, while $k_\rho k_\sigma$ is replaced by $\frac{1}{4}k^2\delta_{\rho\sigma}$. Terms of fourth power in k are found to cancel algebraically each other. Further, the terms linear in v are omitted in (129), since they will vanish on integration.

It is now useful to introduce the new variables

$$P_\lambda=\frac{1}{2}(p_\lambda'+p_\lambda''),\qquad p_\lambda=p_\lambda''-p_\lambda' \tag{130}$$

since the integrand of (129) involves p_λ' and p_λ'' merely as a combined form $p_\lambda'-p_\lambda''$. Then the P-integration can be carried out immediately, yielding $\delta(x'-x'')$. The divergent part of $K(x'-x, x-x'')$ can be isolated by a partial integration of the k^2-terms in the square bracket of (129) with respect to v, as indicated by

$$\int_{-1}^{+1}dv\Big[k^2+m^2u^2\Big\{1+\frac{p^2}{4m^2}(1-v^2)\Big\}\Big]^{-3}$$

$$=2(k^2+m^2u^2)^{-3}-\frac{3}{2}p^2u^2\int_{-1}^{+1}v^2dv\Big[k^2+m^2u^2\Big\{1+\frac{p^2}{4m^2}(1-v^2)\Big\}\Big]^{-4}, \tag{131}$$

then

$$K_\mu(x'-x, x-x'')=\frac{(-2i)^3}{2(2\pi)^8}\delta(x'-x'')\int(dk)(dp)e^{ikp(x-x')}\int_0^1 uduk^2(k^2+m^2u^2)^{-3}$$

$$\times\Big[\beta_\mu\Big\{10-8u-2\eta(1-u)+\frac{p^2}{m^2}\Big(-1+\frac{7}{4}u\Big)\Big\}$$

$$-\frac{i}{m}s_{\mu\nu}\,p_\nu\Big\{2+\frac{u}{2}(9-2\eta)+\beta_\rho{}^2(-4+3u)\Big\}\Big]$$

$$+\frac{(-2i)^3}{(2\pi)^8}\delta(x'-x'')\int(dk)\,(dp)\,e^{ip(x-x')}\int_{-1}^{+1}dv\int_0^1 u\,du\Big[k^2+m^2u^2\Big\{1+\frac{p^2}{4m^2}(1-v^2)\Big\}\Big]^{-3}$$

$$\times\Big[\beta_\mu\Big\{m^2(2-u)^2+p^2(2-u)\Big(1-u+\frac{u^2}{2}\Big)+\frac{p^2}{8}u^2(1-3u)(1-v^2)+\frac{(p^2)^2}{16m^2}u^2(1-u)(1-v^2)\Big\}$$

$$-ims_{\mu\nu}\,p_\nu\Big\{-\frac{u}{2}(2-u)^2+\frac{1}{2}\beta_\rho{}^2u^2(2-u)+u^2(1-u)\,(1-v^2)\Big(\frac{5}{4}-\frac{\beta_\rho{}^2}{2}\Big)$$

$$+\frac{p^2}{8m^2}u^2(\beta_\rho{}^2-u)\,(1-v^2)\Big\}$$

$$-\frac{u^2}{6}\Big\{1+\frac{p^2}{4m^2}(1-v^2)\Big\}\Big\{m^2\beta_\mu u(2\eta-8+\frac{7}{4}\frac{p^2}{m^2})-ims_{\mu\nu}\,p_\nu u\Big(\frac{9}{2}-\eta+2\beta_\rho{}^2\Big)\Big\}\Big]$$

$$-\frac{3(-2i)^3}{8(2\pi)^8}\delta(x'-x'')\int(dk)\,(dp)\,e^{ip(x-x')}\int_{-1}^{+1}dv\int_0^1 u^3\,du\,k^2\Big[k^2+m^2u^2\Big\{1+\frac{p^2}{4m^2}(1-v^2)\Big\}\Big]^{-4}$$

$$\times p^2v^2\Big[\beta_\mu\Big\{10-8u-2\eta(1-u)+\frac{p^2}{m^2}\Big(-1+\frac{7}{4}u\Big)\Big\}$$

$$-\frac{i}{m}s_{\mu\nu}\,p_\nu\Big\{2+\frac{u}{2}(9-2\eta)+\beta_\rho{}^2(-4+3u)\Big\}\Big]. \qquad (132)$$

The integration with respect to k and u may now be performed with the result

$$K_\mu(x'-x,x-x'')=-\frac{2}{(2\pi)^6}\delta(x'-x'')\int(dp)\,e^{ip(x-x')}$$

$$\Big[\beta_\mu\Big\{\Big(\frac{1}{3}(7-\eta)+\frac{1}{12}\frac{p^2}{m^2}\Big)\Big(\log\frac{K+K_0}{m}-\frac{5}{4}\Big)$$

$$+\frac{1}{18}\Big(29-5\eta-\frac{p^2}{m^2}\Big)-\frac{i}{m}s_{\mu\nu}\,p_\nu\Big\{\Big(\frac{5}{2}-\frac{\eta}{3}-\beta_\rho{}^2\Big)\Big(\log\frac{K+K_0}{m}-\frac{5}{4}\Big)+1-\frac{\eta}{9}-\frac{2}{3}\beta_\rho{}^2\Big\}\Big]$$

$$-\frac{2}{(2\pi)^6}\delta(x'-x'')\int(dp)\,e^{ip(x-x')}\int_0^1 dv\Big[\beta_\mu\Big\{4\log\frac{m}{2k_{\min}}+\frac{1}{2}+\frac{1}{24}\frac{p^2}{m^2}$$

$$+\frac{\frac{p^2}{m^2}}{1+\frac{p^2}{4m^2}(1-v^2)}\Big((1+v^2)\log\frac{m}{2k_{\min}}-\frac{19}{48}+\frac{3}{16}v^2\Big)+\Big(4+2\frac{p^2}{m^2}\Big)\frac{\log\Big(1+\frac{p^2}{4m^2}(1-v^2)\Big)}{1+\frac{p^2}{4m^2}(1-v^2)}\Big\}$$

$$-\frac{i}{m}s_{\mu\nu}\,p_\nu\Big\{\frac{\beta_\rho{}^2}{4}-\frac{1}{6}+\frac{1}{1+\frac{p^2}{4m^2}(1-v^2)}\Big(-\frac{19}{24}-\frac{v^2}{12}\Big(\frac{5}{2}-\beta_\rho{}^2\Big)\Big)\Big\}$$

$$-\frac{1}{18}\left\{\beta_\mu\left(2\eta-8+\frac{7}{4}\frac{p^2}{m^2}\right)-\frac{i}{m}s_{\mu\nu}\,p_\nu\left(\frac{9}{2}-\eta+3\beta_\rho^2\right)\right\}\Big]$$

$$-\frac{2}{(2\pi)^6}\delta(x'-x'')\int(dp)\,e^{ip(x-x')}\int_0^1 dv\,\frac{-\frac{p^2}{4m^2}v^2}{1+\frac{p^2}{4m^2}(1-v^2)}$$

$$\times\left[\beta_\mu\left(\frac{7-\eta}{3}+\frac{1}{12}\frac{p^2}{m^2}\right)-\frac{i}{m}s_{\mu\nu}\,p_\nu\left(\frac{5}{2}-\frac{\eta}{3}-\beta_\rho^2\right)\right]. \tag{133}$$

On integration with respect to p and v (if possible) followed by an appropiate rearrangement, (133) becomes

$$K_\mu(x'-x,\,x-x'')=-\frac{1}{2\pi^2}\delta(x'-x'')\beta_\mu\Big[\Big\{\frac{1}{3}(7-\eta)\log\frac{K+K_0}{m}+4\log\frac{m}{2k_{\min}}$$

$$+\frac{1}{36}(\eta-13)\Big\}\delta(x-x')+\frac{1}{12}\left(-\log\frac{K+K_0}{m}+\frac{31}{12}\right)\frac{\Box^2}{m^2}\delta(x-x')+\left(4-2\frac{\Box^2}{m^2}\right)E(x-x')$$

$$+\frac{\Box^2}{m^2}\left\{\left(-\log\frac{m}{2k_{\min}}+\frac{19}{48}\right)(F_0(x-x')+F_1(x-x'))-\frac{1}{12}\left(\eta+\frac{1}{4}\frac{\Box^2}{m^2}\right)F_1(x-x')\right\}\Big]$$

$$+\frac{1}{2\pi^2}\delta(x'-x'')\frac{1}{m}s_{\mu\nu}\frac{\partial}{\partial x_\nu}\Big[\Big\{\left(\frac{5}{2}-\beta_\rho^2-\frac{\eta}{3}\right)\log\frac{K+K_0}{m}-\frac{61}{24}+\frac{13}{36}\eta+\frac{2}{3}\beta_\rho^2\Big\}\delta(x-x')$$

$$-\frac{19}{24}F_0(x-x')-\frac{1}{12}\left(\frac{5}{2}-\beta_\rho^2\right)F_1(x-x')+\left(\frac{5}{2}-\frac{1}{3}\eta-\beta_\rho^2\right)\frac{\Box^2}{4m^2}F_1(x-x')\Big] \tag{134}$$

where $F_n(x)$ is defined by (60) and

$$E(x)=\frac{1}{(2\pi)^4}\int_0^1 dv\int(dp)\,e^{ipx}\,\frac{\log\left(1+\frac{p^2}{4m^2}(1-v^2)\right)}{1+\frac{p^2}{4m^2}(1-v^2)}. \tag{135}$$

Finally, inserting (134) into (115), we obtain

$$\delta j_\mu^c(x)=-\frac{a}{4\pi}\Big\{\frac{1}{3}(7-\eta)\log\frac{K+K_0}{m}+4\log\frac{m}{2k_{\min}}+\frac{1}{36}(\eta-13)\Big\}j_\mu(x)$$

$$-\frac{a}{4\pi}\frac{1}{12}\left(-\log\frac{K+K_0}{m}+\frac{31}{12}\right)\frac{\Box^2}{m^2}j_\mu(x)-\frac{a}{4\pi}\int(dx')j_\mu(x')\Big[\left(4-2\frac{\Box^2}{m^2}\right)E(x-x')$$

$$+\frac{\Box^2}{m^2}\left\{\left(-\log\frac{m}{2k_{\min}}+\frac{19}{48}\right)(F_0(x-x')+F_1(x-x'))-\frac{1}{12}\left(\eta+\frac{1}{4}\frac{\Box^2}{m^2}\right)F_1(x-x')\right\}\Big]$$

$$-\frac{a}{2\pi}\frac{\partial}{\partial x_\mu}\Big[\Big\{\left(\frac{5}{2}-\frac{\eta}{3}\right)\log\frac{K+K_0}{m}-\frac{61}{24}+\frac{13}{36}\eta\Big\}m_{\mu\nu}(x)+\Big\{-\log\frac{K+K_0}{m}+\frac{2}{3}\Big\}m'_{\mu\nu}(x)\Big]$$

$$-\frac{a}{2\pi}\frac{\partial}{\partial x_\nu}\int(dx')\Big[\Big\{-\frac{19}{24}F_0(x-x')-\frac{5}{24}F_1(x-x')+\left(\frac{5}{2}-\frac{\eta}{3}\right)\frac{\Box^2}{4m^2}F_1(x-x')\Big\}m_{\mu\nu}(x')$$

$$+\left\{\frac{1}{12}F_1(x-x')-\frac{\Box^2}{4m^2}F_1(x-x')\right\}m'_{\mu\nu}(x)\Big] \tag{136}$$

in which

$$m_{\mu\nu}(x)=\frac{e}{2m}\psi^\dagger(x)\frac{1}{i}s_{\mu\nu}\psi(x)$$

and

$$m'_{\mu\nu}(x)=\frac{e}{2m}\varphi^\dagger(x)\frac{1}{i}s_{\mu\nu}\beta_\rho^2\psi(x) \tag{137}$$

are both antisymmetric tensors representing electric and magnetic dipole moment densities. The first term of (136) is proportional to the current operator and therefore can be removed by the procedure of charge renormalization. Actually, this is just equal to the current $\delta j_\mu^h(x)$ with opposite sign, as can easily be seen when one compares (136) with (114), and thus it is not necessary at all to renormalize the charge when these terms are added together, quite similarly to the situation encountered in the case of electrons.[14),15)] The remaining parts of (136) represent deviations from the current operator $j_\mu(x)$, induced by the coupling with the vacuum electromagnetic field. They must therefore be observable if they have finite fixed coefficients. In fact, there appear logarithmically divergent factors multiplying $m_{\mu\nu}(x)$ and $m'_{\mu\nu}(x)$ besides $\frac{\Box^2}{m^2}j_\mu(x)$ previously discussed. The former also cannot be removed by the usual technique of renormalization and thus give another example of defects of the present meson theory. This is however a result expected from the beginning, as has been discussed at the end of §2.

(136) has common form both for scalar and vector mesons which are only distinguished by the special representations of β-matrices (10–10 matrix for vector and 5–5 matrix for scalar) and further by γ (which is equal to 4 or 6 for scalar or vector mesons respectively). To interpret (136) literally is, however, fairly dangerous. For example, (136) seems to show that the scalar meson has an anomalous magnetic moment, although its spin angular momentum vanishes. In fact this is not curious at all since the eq. (1) should be regarded as a wave equation of particles with spin 1 lying in the special state described by the 5-rowed representation of the β-matrix.

To discuss the scalar and vector mesons separately, we must go from the Duffin-Kemmer formalism to the usual tensor forms (described by the Klein-Gordon or the Proca equations) by the well-known relations between the components of wave functions. We shall first consider the Klein-Gordon case. In this case, $j'_\mu(x)$ is equal to

$$ie(\partial_\mu\phi^*(x)\cdot\phi(x)-\phi^*(x)\partial_\mu\phi(x)), \tag{138}$$

where $\phi(x)$ and $\phi^*(x)$ are wave functions satisfying the Klein-Gordon equation,

The tensor correspondents of $\partial_\nu m_{\mu\nu}(x)$ and $\partial_\nu m'_{\mu\nu}(x)$ are both easily found equal to that of

$$\frac{1}{4}\frac{\Box^2}{m^2}j_\mu(x). \tag{139}$$

When (114) and (136) are added together, they can be rewritten simply as

$$\begin{aligned}&\delta j_\mu^b(x)+\delta j_\mu^c(x)\\&=\frac{\alpha}{4\pi}\int(dx')j_\mu(x')\Big[\Big(-4+2\frac{\Box^2}{m^2}\Big)E(x-x')\\&\qquad+\log\frac{m}{2k_{\min}}\cdot\frac{\Box^2}{m^2}\Big\{F_0(x-x')+F_1(x-x')\Big\}\Big].\end{aligned} \tag{140}$$

Thus there appears no magnetic moment in the Klein-Gordon formalism. Moreover, the ultra-violet divergences in (136) are those of mere appearance in this case since no such divergence remains in (140).

Our result, (140), is marred only by the fact that there remains yet a logarithmic divergence associated with zero frequency light quanta, the so-called infra-red catastrophe. This is however not discouraging since one knows that the logarithmic dependence on $k_{\min}$ will cancel when all contributions to a certain scattering process are added together as was indicated by the work of Bloch and Nordsieck.[16]

In the Proca case, the current operator $j_\mu(x)$ is equal to

$$ie\{(\partial_\mu\phi_\nu^*(x)-\partial_\nu\phi_\mu^*(x))\phi_\nu(x)-\phi_\nu^*(x)(\partial_\mu\phi_\nu(x)-\partial_\nu\phi_\mu(x))\}, \tag{141}$$

where $\phi_\lambda(x)$ and $\phi_\lambda^*(x)$ are four-potentials of the Proca field. $\partial_\nu m_{\mu\nu}(x)$ and $\partial_\nu m'_{\mu\nu}(x)$ are replaced by

$$\frac{1}{4}\frac{\Box^2}{m^2}j_\mu(x)+\Big(1-\frac{1}{4}\frac{\Box^2}{m^2}\Big)\partial_\nu M_{\mu\nu}(x)+\partial_\nu\partial_\sigma Q_{\mu\nu\sigma}(x) \tag{142}$$

and

$$\frac{1}{2}\frac{\Box^2}{m^2}j_\mu(x)+\Big(\frac{5}{2}-\frac{1}{2}\frac{\Box^2}{m^2}\Big)\partial_\nu M_{\mu\nu}(x)+2\partial_\nu\partial_\sigma Q_{\mu\nu\sigma}(x) \tag{143}$$

where $j_\mu(x)$ is given by (141) and

$$M_{\mu\nu}(x)=\frac{e}{2m}\frac{1}{i}(\phi_\mu^*(x)\phi_\nu(x)-\phi_\nu^*(x)\phi_\mu(x)) \tag{144}$$

$$Q_{\mu\nu\sigma}(x)=\frac{e}{2m^2}\frac{1}{i}(\partial_\mu\phi_\nu^*(x)\phi_\sigma(x)-\phi_\sigma^*(x)\partial_\mu\phi_\nu(x)) \tag{145}$$

are operators representing the dipole and quadrupole moments of the spin 1 field respectively. Inserting (142) and (143) into (136), one obtains the formula of correction to the current in terms of $j_\mu(x)$, $M_{\mu\nu}(x)$ and $Q_{\mu\nu\sigma}(x)$. Since no simplification is attained through this substitution, we shall not give here its form

explicitly. We only remark that $j_\mu(x)$, $M_{\mu\nu}(x)$, and $Q_{\mu\nu\sigma}(x)$ are all multiplied with divergent coefficients.

§ 7. Discussion and summary

In the preceding three sections we have analysed in detail the reactive effects arising from the interaction between meson and electromagnetic fields in the second order approximation. The aim of this rather lengthy and laborious investigation is, as was mentioned in the Introduction, more of an academic nature than practical. The obtained results even seem to accentuate this situation, since the theoretically observable correction to the original mesonic charge-current has turned out to be generally divergent and consequently failed to furnish us with quantitative conclusions about the electromagnetic properties of meson field, at least in the case of vector type. It is somewhat discouraging to find out that the Tomonaga-Schwinger theory, which brought about so great a success in the quantum electrodynamics of electrons cannot claim its range of efficacy to be very wide. But this is not unexpected. We know that the electron field with its Fermi statistics and first order character of the equation of motion is most favorable since the singularities of interaction are minimized. We should not expect the same thing from meson fields which obey Bose statistics and second order wave equations. There may lie in this fact a deeper reason which is more than mathematical, though we cannot express it in a convincing way.

But apart from quantitative considerations, we have obtained some qualitative and formal conclusions which are rather important. The program of quantum electrodynamics developed by Tomonaga, Schwinger, Feynman and Dyson can also be extended to the present case without essential modifications, at least from a formalistic point of view. However, a little more precaution and labor as well as minor deviations are necessary. First of all, the integrability requirements of the interaction representation necessitates us to supplement the original interaction Hamiltonian with a suitable additional term. This causes a peculiar trouble when we insert a counter-self-energy term in the interaction, since the "mass" of the meson field is not an integrable quantity by itself and does not commute with the charge-current either, in the sense of Duffin-Kemmer formalism. We therefore can not bring about the interaction representation with counter-term from the original Schroedinger representation by two steps of transformation, namely first eliminating the free field and next attaching the counter term, as was the case for electron, but have to do it in one step, thereby using for the free field an experimental mass instead of a mechanical mass. A further complication arises from the fact that there are two scalar quantities (1 and β_p^s) built up of spin matrices so that we have to adopt both of them in order to eliminate self-energy effects. This was avoided by making use of a relation by which they are connected together with a divergence expression which can be counted upon as

vanishing for free field. Translated in tensor formalism, the above fact is reflected in the appearance of terms quadratic in ϕ and $\partial_\mu\phi$. The latter, however, can be identified with the former by a partial integration, or alternatively, by a renormalization of the Lagrangian.

The cumbersome procedure of using correct integrable form of the interaction Hamiltonian with mass-term which is not yet given explicitly, is then replaced by a simpler one. The latter method consists in an appropriate definition of Dyson's ordering operator P when two points under consideration lie in a space-like relation with an infinitesimal distance. In actual expression, we have only to adopt the interaction Lagrangian density itself as the interaction operator and use the corrected Green function, the first term of (23), which represents the vacuum expectation value of ordered field operators. This has been suggested by several authors,[8] and is very interesting as well as instructive since it offers us possibilities for a consistent extension of the present localizable theory to a non-localizable one where the old Hamiltonian description would lose its validity.

Making use of these manipulations, the program of mass and charge renormalizations has been effected without difficulty, and the observable deviation from the original current has been determined. The charge renormalization is required only for that part which describes the polarization of the meson field due to an external electromagnetic field, as is the case for electron, at least in the e^2-approximation. When we go to higher order corrections, this may not hold in general.

The actual result in the present case has turned out to be divergent. But for scalar meson this divergence is only superficial, since we obtain a convergent expression when we go back to the tensor formalism. Further, it has no spin magnetic moment in spite of its seeming appearance in the Duffin-Kemmer representation. For vector meson, on the other hand, we have essential divergences which cannot be removed. Moreover, there appears a true anomalous magnetic moment as well as a term which may be interpreted as a quadrupole moment. The latter did not happen in the electron because of the simpler structure of the Dirac equation. All these terms, however, are multiplied by divergent factors, which makes it difficult to come to a conclusion about experimental verification of such effects. To relieve ourselves of this point, various devices may be suggested. For example, we may start with an expression for zeroth order current in which all these anomalous terms are contained with unknown factors yet to be determined so that all divergence may cancel out in the final result. This, however, fails since such an inclusion of anomalous terms give rise to still higher divergences in the next step, as was revealed by Feldman,[9] so that it must be continued *ad infinitum* until at last we reach a meaningless, or at best an integral rather than an differential, expression for the current. Another more effective means is furnishd by the Pauli regulator, or its more realistic version of Umezawa[17] and Feldman. With suitable conditions, we can

certainly scratch all undesirable terms from the result. But again we shall have no excuse for its highly arbitrary character, so long as we can give no further theoretical or experimental foundation about it. Therefore we shall be here content only with pointing out these difficulties as they are, hoping that this may help towards their correct and thorough solution.

The prospect over any application of the present result to practical problems is thus rather poor. As is well known by Christy and Kusaka's work on burst production the interaction between vector meson and radiation field becomes increasingly large at high energies, yielding an essential difference from scalar meson.[18] But at these energies, the higher order corrections are in general also large, and we may expect that the interaction does not lead to a divergent catastrophe at extreme relativistic region. In reality such corrections are by themselves divergent, and no definite refinement of Christy-Kusaka's results seems to be possible for vector mesons.

We have, however, more conservative ways of analysis, as will be seen below. Terms are divergent, but we know their physical properties as well as the effects they would cause if the infinite factors multiplied on them were actually finite. The order of magnitude of these "finite" coefficients may be determined by analogy to other really convergent cases. In some cases we could even predict their sign from the divergent coefficients. With such convention, we get for the charge-current of a vector meson a picture which consists of an elliptically extended intrinsic charge-current and a spinning current. The former includes a quadrupole moment of the vector meson. It would be interesting if such effects could really be observed. Their magnitude, however, would be too small, and the vector meson itself too short-lived, if exists at all. Hence the only possibility may be found in the deuteron, in which vector mesons might play a role and contribute a magnetic and a qurdrupole moment. But it may be premature to enter into such minute effects in the present stage where more basic facts about the deuteron remain to be explained. These anomalous terms of the meson would also cause an anomalous behavior of a single nucleon with which it interacts. But it would again be a higher order effect, even if it turned out convergent. In this connection it may be noted that a single nucleon cannot have an anomalous quadrupole moment so long as it obeys the Dirac equation.

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On the Problem of Gauge-Invariancy and Divergence in the Theory of Elementary Particles, I

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§ 1. Introduction

Previously, we[1] synthetically analysed the problem of vacuum polarization by considering simultaneously various charged particles. By this analysis, it becomes clear that the problem of gauge-invariancy of vacuum polarization closely connects with the definition of the vacuum in quantum field theory.[2] It is the purpose of this paper to analyse this problem of the definition of the vacuum under more comprehensive version and to investigate the problem of gauge-invariancy and various divergencies in the present quantum field theory. In order to make clear the physical meaning, correspondence theoretical argument is used in this paper. This argument provides some interesting suggestions on the nature of various difficulties of present quantum field theory. Thus we will propose a correspondence theoretical argument relating to the vacuum and expect in future its complete formulation in quantum field theory.

Recently two important progresses were made in the region of quantum electrodynamics, one of which is the completely relativistic formulation of the theory,[3][4] and the other the classification of types of divergencies on the basis of this formalism.[4][5] It was clarified by means of the above mentioned method (or by the improvement of usual perturbation calculation) that the self-energy of an electron has the transformation property of the mass and that no diverging term except the mass-renormalization term and the charge renormalization term appears in the system of electrons and electromagnetic fields. On the basis of this fact, the renormalization method (Self-Consistent Subtraction Method) was proposed in order to evade the divergence difficulties and to obtain the finite terms to be compared with the experiments.[4][5] The essential point of this method is that one may put divergencies of the mass type and of the charge type into the observed mass and charge of electron respectively, as the alterations of the mass and the charge are, if they are finite, to be already included in the observed mass and charge. This method led, in fact, to the Lamb-Retherford shift of the energy level of hydrogen atom which agrees almost with the experiment. However, there still remain many problems to be investigated. It must be, at first, investigated whether this renormalization method is valid

not only for the system consisted of electrons and electromagnetic fields, but also for more general system. More important problem is to analyse the character of divergence difficulties evaded in this method. It is demanded to make efforts to disclose the internal conflict in the present theory of elementary particles in the course of its adjustment of various difficulties and to grasp the correct laws of motion in future. Especially, various divergence-difficulties appearing in the theory of elementary particles seem to be related with the structure of elementary particles.

As the first step of the analysis[6] of the structure of elementary particles, hitherto, we have had Prof. Sakata's view point, who suggested the synthetical studies on the quantum field theory by taking account of various fields simultaneously. The theories[1)2)3)] developed in this direction have revealed various features in the theory of interactions between elementary particles. From these "concrete" theories, it may be possible to construct "abstract" theories by abstracting each of these features. (Of course such a abstract theory is frequently of transient character but, by means of this theory, we may be able to investigate the difficulties from the more extensive standpoint). For instance, the one aspect of contents of C-meson theory for solving the divergence-difficulties of the self-energy of an electron which has the transformation property of mass term was abstracted into the self-consistent subtraction method.[7] Such a theory, that was established by abstracting only one feature of a concrete theory, however, cannot explain all difficulties. The renormalization method fails, in fact, in the problem of the self-stress of an electron.[8] Through the way to construct many abstract theories by extracting various contents of the concrete theory, we can make clear up the physical character of these difficulties. Recently, according to such a method of analysis, we mentioned that one characteristic of mixture of charged particle field which we[1] and also Feldman[9] and Rayski[10)11)] introduced in the region of the vacuum polarization, leads to Pauli's regularization method and the other aspect is related to the problem of the definition of the vacuum.[2] In this paper we will discuss in details this second point and show that it is possible to perform some new important analysis on the problem of gauge-invariancy and divergencies from wider field of view based on this argument relating to the vacuum. Then, we will propose this argument on the vacuum in quantum field theory, independently of validity of the mixture-theory of various charged particles[1] from which we started. In this paper, this argument is developed correspondence theoretically and we will in future expect its complete formulation in quantum field theory.

The mass term of a photon destroys the condition of gauge-invariance which must be satisfied in quantum electrodynamics. Accordingly we can expect that it vanishes in future correct theory. In the case of a neutral vector field with mass $\varkappa(\varkappa \neq 0)$, if its self-energy have a term which does not vanish when $\varkappa \to 0$, this term may vanish as in the case of the mass term of a photon in future correct

theory.* Thus, in the case of various fields, non-zero mass terms which must vanish in future correct theory may remain in the calculation of present field theory. We may call such terms as "generalized non-gauge-invariant terms" ((g.n.g.) terms). The (g.n.g.) term may appear in all problems which relate to the vacuum, e.g. the problem of self-energies of elementary particles, the problem of meson decay through the medium of vacuum particles, etc.

In this paper, the correspondence theoretical method is proposed which leads to vanishing (g.n.g.) terms. In § 2, We will investigate the physical origin of the self-energy of a photon and give the method in which it vanishes. In following chapter, by means of this method, problems of self-energies of elementary particles and meson decay** through the medium of vacuum particles are investigated.

§ 2. The self-energy of a photon

Denoting the self-energies of a photon with the momentum $\boldsymbol{l}$ due to various charged fields by W_l, the mass term of photon is appeared in the Hamiltonian as $\frac{W_l}{4\pi}\boldsymbol{A}^2$; so that, if W_l is not zero, it destroys the condition of gauge-invariance. Previously, we had calculated the self-energies of a photon due to the various charged fields (mass μ) and obtained the following results[1]:

$$W_l=\frac{ne^2}{3\pi l}\left[\frac{1}{2}w-\mu^2\right], \qquad l\equiv|\boldsymbol{l}|, \tag{1}$$

$$n=\begin{cases}+1 & \text{for scalar charged field,}\\ -2 & \text{for spinor charged field,}\\ +3 & \text{for vector charged field.}\end{cases} \tag{2}$$

From these results we[1] had proposed the following fields-mixture for eliminating the self-energy of a photon: m scalar, n spinor and l vector fields with masses μ^s, μ^e and μ^v respectively:

$$\left.\begin{aligned}&-2n+m+3l=0,\\ &-2\sum_{i=1}^{n}(\mu_i^e)^2+\sum_{i=1}^{m}(\mu_i^s)^2+3\sum_{i=1}^{l}(\mu_i^v)^2=0.\end{aligned}\right\} \tag{3}$$

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* In the case of neutral scalar field, too, by means of introducing mass coordinate in the five dimensional representation, the condition of gauge-invariance can be introduced.
In this case we must care about the essential difference between the electromagnetic field and the neutral vector field with non-vanishing mass. cf. F. J. Belinfante, Phys. Rev. **75** (1949), 337.

** In this paper, system of unit $\hbar=c=1$ is used. The spatial component of four dimensional vector $k(k_1, k_2, k_3, k_4=ik_0)$ is represented by bold type letter $\boldsymbol{k}$. $(k^{(1)}, k^{(2)})$ and $(\boldsymbol{k}^{(1)}, \boldsymbol{k}^{(2)})$ means scalar products in four dimensional space and three dimensional space respectively.

These conditions for mixture fields have two interesting characteristics. One is that the second formula of (3) is the relation between masses. This feature gives the realistic meaning for Pauli's regularization method. The other important aspect of this mixture theory, is that the self-energies of a photon due to various charged fields are in the same proportion and of the same signs as the zero-point energies of these charged particles:

$$E_{\text{zero}}=n\int\sqrt{p^2+\mu^2}\,\frac{d\boldsymbol{P}}{(2\pi)^3}, \tag{4}$$

where n has the same meaning as in Eq (2). It gives some suggestions for the problem of gauge-invariancy to abstract this aspect of affairs from the mixture theory and to analyse it.

In order to carry out such a analysis, we return to Dirac-Heisenberg-Weisskopf's positron theory. Dirac[3] and Heisenberg[4] used the density matrix R_s and the Hartree approximation in their calculations. When $\psi(x)$ means the wave function of the electron, R_s is:

$$(x'|R_s|x'')=\frac{1}{2}\{\sum_{\text{occ.}}\psi_n^*(x')\psi_n(x'')-\sum_{\text{unocc.}}\psi_n^*(x')\psi_n(x'')\}, \tag{5}$$

$\sum_{\text{occ.}}$ or $\sum_{\text{unocc.}}$ means the summation over the states which are occupied or unoccupied by the electron, respectively. They assumed, that the nonsingular quantity, which is obtained by subtracting the singular quantity S from the R_s with singularity, is the actually observed quantity: where S is determined such as $R_s-S=0$ in vacuum. Therefore, S is the charge-density in vacuum: consequently, the determination of S is equivalent to defining the vacuum. Dirac and Heisenberg obtained the following result for S:

$$(x'|S|x'')=u\frac{a^2x_\rho}{(x^\lambda x_\lambda)^2}-\frac{v}{(x^\lambda x_\lambda)}+w\log\frac{|x^\lambda x_\lambda|}{C}\quad(x=x'-x''), \tag{6}$$

u, v and w are the quantities without singularities. Especially,

$$u=-\frac{i}{(2\pi)^2}\exp\left(-ie\int_{x'}^{x''}A_\lambda dx_\lambda\right) \tag{7}$$

in which the integral means the line integral along the straight line joining two points x and x'.

Thus, Dirac and Heisenberg constructed the gauge-invariant positron theory. The physical meaning of this subtraction theory was given by Weisskopf.[5] According to Weisskopf, to subtract (6) from R_s is equivalent to following subtractions:

(1) Subtraction of the energy of the vacuum electrons in free space.

(2) Subtraction of the charge and current density of the vacuum electrons in free space.

(3) Subtraction of the chronologically and spatially constant polarizability of vacuum.

Subtraction (3) corresponds to the charge renormalization, and (1) and (2) to the definition of the vacuum. Then according to Weisskopf, however, it is not adequate to use the ordinary zero-point energy as the energy of vacuum electrons. The factor u appearing in (6) suggests that one has to use the following quantities as the energy of the vacuum:

$$U_0^{(e)}=-\frac{2}{(2\pi)^3}\int\{\sqrt{(\boldsymbol{P}-e\boldsymbol{A})^2+\mu^2}-eA_0\}d\boldsymbol{P}. \tag{8}$$

The factor u, as its form shows, is important for obtaining the gauge-invariant result. Accordingly, the definition (8) of the energy of the vacuum is essential for constructing the gauge-invariant theory. From the definition (8) of the energy of the vacuum we get following postulate:

(I) As the self-energy of the vacuum we must take one in which p_μ is replaced by $p_\mu-eA_\mu$ instead of the ordinary zero-point energy; more generally we must define the vacuum with $p_\mu-eA_\mu$ and not with p_μ. But to define the vacuum by (8) is only the correspondence theoretical argument, without considering the second quantization of electromagnetic field. Accordingly we will investigate the physical meaning of (I) by using the classical theory.

In classical dynamics, for the charged particle with momentum $\boldsymbol{P}$ and velocity v, we have following relation:

$$\sqrt{(\boldsymbol{P}-e\boldsymbol{A})^2+\mu^2}-eA_0=\frac{\mu}{\sqrt{1-v^2}}, \tag{9}$$

i.e., postulate (I) specifies the state of charged particle by its velocity v. More interesting fact is that since the right side of (9) has the transformation property of the fourth component of the energy-momentum four vector, $\sqrt{p^2+\mu^2}$ has not such a transformation property; accordingly, it is natural that in the usual theory defining the energy of the vacuum by ordinary zero-point energy, only the spatial component of the electromagnetic potential contributes to the mass term of a photon and gives therefore the relativistically non-invariant result, and also that the usual definition on the vacuum (4) provides non-gauge-invariant result, because in Dirac-Heisenberg's gauge-invariant subtraction theory, (8) is used as the energy of the vacuum.

It is also possible to make clearer the relation between the self-energy of a photon and the definition of the vacuum. By means of postulate (I), the energies of the vacuum of various charged fields are defined as follows:

$$U_0=\frac{n}{(2\pi)^3}\int\{\sqrt{(\boldsymbol{P}-e\boldsymbol{A})^2+\mu^2}-eA_0\}d\boldsymbol{P}. \tag{10}$$

Expanding (10) into power series for the charge e, as in the perturbation calculation, and taking the terms up to e^2, we obtain[2]:

$$\left.\begin{aligned} U_0&=(E_{\text{zero}}-eA_0)+U_2\cdots\cdots,\\ U_2&=\frac{ne^2}{3\pi}\left[\frac{1}{2}w-\mu^2\right]\frac{\boldsymbol{A}^2}{4\pi}, \end{aligned}\right\} \tag{11}$$

$$w\equiv(P,\,P')=2\boldsymbol{P}^2+\mu^2=E_P^2-(\boldsymbol{P},-\boldsymbol{P}). \tag{11'}$$

U_2 is just the mass term (1) of a photon. When we use, therefore, (10) as the energy of the vacuum to be subtracted in the calculation, the self-energy of a photon vanishes and gauge-invariant vacuum polarization can be obtained.*

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* It is well known that the value of the self-energy of a photon is very ambiguous and depends on the method of calculation. The result (1) is obtained when integration over energy is firstly performed in the calculation in four-dimensional energy-momentum space. This definition of integration calculation gives (4) as zero-point energies. Therefore, it is adequate to compare (11) with (1).

Previously we pointed out[9][16][17] that above-mentioned relation between zero-point energies of an electron (that is $W\infty n$) is established independent of definition of integration calculation.

Pauli[12] and Wentzel[8] calculated the self-energies of a photon by using the integral representation of Δ-function.

$$\Delta^{(1)}(x)=\frac{1}{(2\pi)^3}\int(dk)\int d\alpha e^{i\alpha(k_\mu^2+\mu^2)+ikx}. \tag{i}$$

Then, we obtain as the self-energies W of a photon due to various charged particles by using (i):

$$W=\frac{ne^2}{8\pi}\int\frac{dw}{w}\,\varepsilon(w)\left[i\frac{w^2\varepsilon(w)}{\pi^2\mu^4}(2C_{\mu\nu}-C_{\lambda\lambda}\delta_{\mu\nu})+\mu^2\delta_{\mu\nu}\right]e^{iw\frac{A^\mu A^\nu}{(4\pi)}}\quad\left(\varepsilon(w)\equiv\frac{|w|}{w}\right), \tag{ii}$$

where

$$C_{\mu\nu}\equiv\int k_\mu k_\nu e^{i\frac{w}{\mu_2}k_\mu^2}(dk). \tag{iii}$$

$C_{\mu\nu}$ is ambiguous integral. Pauli and Wentzel obtained finite value of the self-energy of a photon by means of following definition of $C_{\mu\nu}$ which is obtained from (iii) by partial integration:

$$C_{\mu\nu}=-\delta_{\mu\nu}\frac{i\mu^2}{2w}\int e^{i\frac{w}{\mu_2}(k_\mu^2+\mu^2)}(dk) \tag{iv}$$

$$=-\delta_{\mu\nu}\frac{\pi^2\mu^6}{2w^3}\varepsilon(w)e^{iw}, \tag{v}$$

$$\left(\int(dk)e^{i\alpha k_\mu^2}=-\frac{i\pi^2}{\alpha^2}\varepsilon(\alpha)\right).$$

Then,

$$W=-\frac{n\mu^2}{4\pi}\,\frac{e^2A_\mu^2}{(4\pi)}. \tag{vi}$$

For $n=-2$ we get $W=\frac{e^2\mu^2}{8\pi^2}A_\mu^2$ as the mass term of a photon due to electron which is the result obtained by Pauli and Wentzel.

We will show that the postulate (1) make self-energy of a photon zero also in the case of the above definition of integration calculation. The zero-point energy E_{zero} of scalar charged field U is represented by using Δ-function as follows:

$$\begin{aligned} E^s_{\text{zero}}&=\frac{1}{4\pi}\int\left\langle U(x)\left\{\frac{\partial^2}{\partial x_4{}^2}-\frac{\partial^2}{\partial x_4{}^2}+\mu^2\right\}U(x)\right\rangle_0 dx\\ &=\frac{1}{4\pi}\int\delta(x-x')\left\{\frac{\partial^2}{\partial x_4{}^2}-\frac{\partial^2}{\partial x_4{}^2}+\mu^2\right\}\Delta^{(1)}(x-x')dx\\ &=-\frac{2}{(2\pi)^34\pi}\int(dk)k_4{}^2\delta(k\mu^2+\mu^2) \end{aligned}$$

$$=-\frac{2}{(2\pi)^3 4\pi}\int da\int(dk)k_4{}^2e^{ia(k_\mu{}^2+\mu^2)}.$$

Similarly, zero-point energies E_{zero} of various charged fields are as follows:

$$E_{\text{zero}}=\frac{-2n}{(2\pi)^3 4\pi}\int da\int(dk)k_4{}^2e^{ia(k_\mu{}^2+\mu^2)}. \tag{vii}$$

By partial integration, as in the case of (iv), we get:

$$E_{\text{zero}}=\frac{-in}{(2\pi)^3 4\pi}\int(dk)\int da\frac{1}{a}e^{ia(k_\mu{}^2+\mu^2)}. \tag{viii}$$

According to the postulate (I), we obtain U_0 as the energy of the vacuum by replacing k_μ in (viii) by $k_\mu-eA_\mu$. By expanding U_0 into power series of e, e-term vanishes by integration calculation and we get e^2-term U_2 as follows:

$$\begin{aligned}U_2&=-\frac{ine^2}{(2\pi)^3 4\pi}\int(dk)\int\frac{da}{a}\{ia A_\mu{}^2-2a^2(Ak)^2\}e^{ia(k_\mu{}^2+\mu^2)}\\&=\frac{ine^2}{(2\pi)^3 4\pi}\int(dk)\int\frac{da}{a}\{2a^2k_\mu k_\nu-a^2(k_\lambda{}^2+\mu^2)\delta_{\mu\nu}\}e^{ia(k_\mu{}^2+\mu^2)}A_\mu A_\nu.\end{aligned} \tag{ix}$$

By using (viii) and transforming $a\to\frac{w}{\mu^2}$, we obtain U_2 as follows;

$$\begin{aligned}U_2&=\frac{ne^2}{8\pi}\int\frac{dw}{w}\varepsilon(w)\left\{\frac{iw^2\varepsilon(w)}{\pi^2\mu^4}(2C_{\mu\nu}-C_{\lambda\lambda}\delta_{\mu\nu})+\mu^2\delta_{\mu\nu}\right\}e^{iw}\frac{A_\mu A_\nu}{(4\pi)}\\&=-\frac{n\mu^2}{4\pi^2}\frac{e^2A_\mu{}^2}{(4\pi)},\end{aligned} \tag{x}$$

which agrees perfectly with (ii). Therefore, the self-energies of a photon can be made to vanish by postulate (I). On the other hand, Schwinger[4] and Furry[9] obtained the zero self-energy of a photon by means of suitable definition of integration calculation. We can show that their definition of calculation is very one that leads to vanishing e^2A^2-term (the mass terms of a photon), which comes from the energies of the vacuum defined according to postulate (I).

From (4) zero-point energies are:

$$E_{\text{zero}}=-\frac{2n}{(2\pi)^3(4\pi)}\frac{1}{4}\int(dk)k_\mu{}^2\delta(k_\mu{}^2+\mu^2).$$

According to pestulate (I) we obtain the energies of the vacuum by replacing k_μ by $k_\mu-eA_\mu$ in E_{zero}.

$$\delta((k_\mu-eA_\mu)^2+\mu^2)=\delta(k_\mu{}^2+\mu^2)-eA_\nu\delta_\nu(k_\mu{}^2+\mu^2)+\frac{e^2}{2}A_\nu A_\mu\delta_{\mu\nu}(k_\mu{}^2+\mu^2)+\cdots\cdots, \tag{xi}$$

where

$$\delta_\nu(k_\mu{}^2+\mu^2)\equiv\frac{\partial}{\partial k_\nu}\delta(k_\mu{}^2+\mu^2),$$

$$\delta_{\mu\nu}(k_\mu{}^2+\mu^2)\equiv\frac{\partial^2}{\partial k_\mu\partial k_\nu}\delta(k_\mu{}^2+\mu^2),$$

$$\delta'(x)=\frac{\partial\delta(x)}{\partial x}.$$

The e^2-term U_2 of U_0 is, by means of (xi) as follows:

$$\begin{aligned}U_2=-\frac{2ne^2}{(2\pi)^3 4}\int(dk)[&2A_\mu A_\nu k_\mu\delta_\nu(k_\mu{}^2+\mu^2)\\&+\frac{e^2}{2}k_\lambda{}^2A_\nu A_\mu\delta_{\mu\nu}(k_\mu{}^2+\mu^2)+A_\mu{}^2\delta(k_\mu{}^2+\mu^2)]\frac{1}{(4\pi)}.\end{aligned}$$

By partial integration, we obtain:

$$U_2=\frac{ne^2}{(2\pi)^3 4}\int(dk)[2\delta(k_\mu{}^2+\mu^2)\delta_{\mu\nu}+4k_\mu k_\nu\delta'(k_\mu{}^2+\mu^2)]\frac{A_\mu A_\nu}{(4\pi)}. \tag{xii}$$

(xii) agrees perfectly with the mass terms of a photon obtained by Schwinger and Furry. They obtained vanishing self-energy of a photon from (xii) by partial integration.

However, postulate (I) is the correspondence theoretical definition and so we cannot directly proceed this method in present form into the quantized field theory. A photon, introduced into the charged particle fields, make increase the zero-point energy of every charged particle up to E_{zero} (because E_{zero} is defined by means of momentum of charged particle in interaction). So that, we rewrite the postulate (I) as follows:

(I′) **The zero-point energy is deviated from the energy of the vacuum on account of the existence of interaction.**

This definition of the vacuum is required in the calculation for interacting field and (I′) requires to define the vacuum in considering the existence of the interaction.*

In order to perform the correspondence theoretical investigations in various cases, moreover, we set following postulate:[20]

(II′) **The deviation of zero-point energy from the energy of the vacuum is determined by means of the deviation of the equation of motion of interacting field from that of free case.**

Postulates (I′) and (II′) are reduced to the postulate (I) in the problem of vacuum polarization in quantum electrodynamics.**

§ 3. The problem of gauge-invariancy

We will study the more general problem of gauge-invariancy than that of the self-energy of a photon.

a) *System of charged field and electromagnetic field*

We obtain following terms as the energy of the vacuum by the strict integration of (10);

$$U_0 = \frac{n}{(2\pi)^3}\int[\{(\boldsymbol{P}-e\boldsymbol{A})^2+\mu^2\}^{\frac{1}{2}} - eA_0]d\boldsymbol{p}$$

$$= \frac{n}{(2\pi)^3}\int d\boldsymbol{p}\sqrt{\boldsymbol{p}^2+\mu^2} + \frac{n}{2\pi}\left(\frac{w}{2}-\mu^2\right)\frac{e^2\boldsymbol{A}^2}{4\pi} - \frac{n}{2\times 60\pi}e^4(\boldsymbol{AA})(\boldsymbol{AA}). \qquad (10')$$

The second term is of course the mass term of a photon (11).

The third is the term which contributes to the process of photon by photon

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* This fact seems to show that we cannot define the absolute vacuum independent of the interaction, but must consider about the concept of vacuum in considering the interaction. Since the relativistic covariance and the gauge-invariance of the theory of the system consisting of interacting fields is established in considering the interaction, customary definition of the vacuum independent of the interaction may destroy the various invariancies of results of calculations in problems for interacting fields.

** Mr. Machida proposed more quantum theoretical method of treating these postulates (I′) and (II′). In his argument, the field operator in vacuum is the stationary-state-solution with fixed $\not{p}\mu$ (momentum and energy) of its wave equation including interaction.

scattering. This term perfectly agrees with the non-gauge-invariant term appearing in the case of the photon by photon scattering through electron pairs ($n=-2$) in perturbation theory which Euler found.[20] Therefore, in this case, we can obtain gauge-invariant result by means of postulates (I′) and (II′).

No terms of higher order than the fourth in charge (terms ∞c^m, $m>4$) appear in (10′).*

b) *Decay of a neutral meson into photons through the charged spinor particle pairs*

As the interaction between charged spinor particle ψ (mass μ) and neutral scalar meson φ (mass $\varkappa$) we take the form:

$$f\psi^*\beta\psi\varphi. \tag{12}$$

Then, according to the ordinary perturbation calculation in the momentum space, we obtain the following term as the non-gauge-invariant term:[21]

$$\frac{\mu e^2}{6\pi^2} f\varphi \boldsymbol{A}^2. \tag{13}$$

But in this case, we define the energy of the vacuum by using (I′) and (II′) instead of E_{zero}. According to (II′) we write the equation of motion of ψ as follows:

$$\{(\alpha, \boldsymbol{p}-e\boldsymbol{A})+\beta\mu+f\beta\varphi\}\psi=0. \tag{14}$$

This is just the free field equation in which μ is replaced by $\mu+f\varphi$. According to (I′), as the energy of vacuum we take

$$U_0^{(e)}=-\frac{2}{(2\pi)^3}\int\{\sqrt{(\boldsymbol{P}-e\boldsymbol{A})^2+(\mu+f\varphi)^2}-eA_0\}d\boldsymbol{p}. \tag{15}$$

Expanding into power series of e and f and taking the term of the order e^2f, this term agrees completely with (13). Therefore, defining the energy of vacuum by (15) according to postulates (I′) and (II′), the gauge-invariant result can be obtained also in this case.

When a neutral vector meson U_μ decays through the nucleon pairs by the interaction $f\psi^+\gamma_\mu\psi U_\mu$: it is suitable to define the energy of vacuum as E_{zero} in which p_μ is replaced by

$$p_\mu-eA_\mu-fU_\mu. \tag{15′}$$

Expanding this energy of vacuum, e^2f-terms vanish by performing the integration over angle, so that, terms corresponding to the case of neutral vector meson decaying into two photons disappear. Term of order e^3f is $\frac{e^3f}{15\pi^2}(\boldsymbol{A}^2)(\boldsymbol{AU})$ which is

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* Accordingly, in the case of the mixture theory (3) or Pauli's regularization method, we may use ordinary zero-point-energy E_{zero} as the definition of the energy of the vacuum.

just non-gauge-invariant term in the case of neutral vector meson decaying into three photons. Thus, it becomes obvious that the problem of gauge-invariancy connects with the definition of the vacuum.

§ 4. The self-energies of elementary particles

The postulates (I′) and (II′) would lead to different results from usual ones in problems connecting with the subtraction of vacuum. One of such a problem is that of self-energies of elementary particles. In this case (I′) and (II′) have following physical meaning: the self-energy of elementary particle a due to the field b is the additional energy to the vacuum that arises when a is brought into the vacuum of b. This consists of the zero-point-energy $E_{a,0}$ which comes from the zero-point oscillation of b forced by a, and of the increment of energy of a due to b. In order to subtract the deviation of energy of a due to the forced zero-point oscillation of b, one must use the postulates (I′) and (II′). It was the most simple example that the self-energy of a photon vanished by means of (I′) and (II′). Now, we reexamine the self-energies of the neutral mesons and of the charged particles by using (I′) and (II′).

(a) The self-energy of neutral scalar meson (C-meson)

(i) *Self-energy due to the charged spinor field (mass μ)**

Adopting (12) as the interaction, the mass term of neutral scalar meson (mass $\varkappa$) is

$$W_{\text{C.F.}}=-\left\{\frac{f^2}{\pi}p^2+\frac{f^2}{\pi}\left(\frac{\varkappa^2}{2}-3\mu^2\right)\log\frac{2p}{\mu}+\text{finite term}\right\}_{p\to\infty}\frac{\varphi^2}{(4\pi)} \tag{16}$$

in usual perturbation calculation.[1] On the other hand, the energy of vacuum in this case is defined by (15) according to (I′) and (II′). Expanding it into power series of e and f, we obtain as the f^2-term

$$-\left\{\frac{f^2}{\pi}p\sqrt{p^2+\mu^2}-\frac{3\mu^2f^2}{\pi}\log\frac{2p}{\mu}+2\mu^2f^2\right\}_{p\to\infty}\frac{\varphi^2}{(4\pi)}. \tag{17}$$

This agrees completely with $W_{\text{C.F.}}$ ($\varkappa=0$) including the finite term.
The condition $\varkappa=0$ seems to show that $W_{\text{C.F.}}(\varkappa=0)$ is (g. i. g.) term.
Thus, when defining the energy of vacuum also in the case of $\varkappa\neq0$ by Eq. (15), the mass term $W'_{\text{C.F.}}$ of neutral scalar meson due to the charged spinor field is by subtraction (17) from (16)

$$W'_{\text{C.F.}}=-\left\{\frac{\varkappa^2f^2}{2}\log\frac{2p}{\mu}+\text{finite term.}\right\}_{p\to\infty}\frac{\varphi^2}{(4\pi)}, \tag{18}$$

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* In following, we designate the mass of charged particle as μ and that of neutral particle as $\varkappa$.

$$W'_{\text{C.F.}}(\varkappa=0)=0, \tag{18'}$$*

which has only logarithmic divergence.**

(ii) *Self-energy due to the charged scalar field U(mass μ)*

We take following form as interaction:

$$-\frac{f\mu}{4\pi}U^*U\varphi-\frac{g^2}{4\pi}U^*U\varphi^2, \tag{19}$$

then the mass term $W_{\text{C.S.}}$ of neutral scalar meson is[1]

$$W_{\text{C.S.}}=\left\{-\frac{g^2}{2\pi}p\sqrt{p^2+\mu^2}+\frac{\mu^2}{\pi}\left(\frac{g^2}{2}-\frac{f^2}{4}\right)\log\frac{2p}{\mu}+(f.t.)\right\}_{p\to\infty}\frac{\varphi^2}{(4\pi)}, \tag{20}$$

$$(f.t.)=\frac{f^2}{2}\frac{\mu^2}{\pi\varkappa}\sqrt{\mu^2-\frac{\varkappa^2}{4}}\left\{\sin^{-1}\frac{1}{\mu}\left(\mu^2-\frac{\varkappa^2}{4}\right)^{\frac{1}{2}}-\sin^{-1}\mu\left(\mu^2-\frac{\varkappa^2}{4}\right)^{-\frac{1}{2}}\right\} \tag{20'}$$

according to usual perturbation calculation. While, the equation of motion of the U-field is

$$\{\Box-\mu^2+\mu f\varphi+g^2\varphi^2\}U=0, \tag{21}$$

i.e., one that μ^2 is replaced by $(\mu^2-\mu f\varphi-g^2\varphi^2)$ in free U-field equation. Then, according to (I') and (II'), we define the energy of vacuum of U-field as

$$\frac{1}{(2\pi)^3}\int\{\sqrt{(\boldsymbol{P}-e\boldsymbol{A})^2+\mu^2-\mu f\varphi-g^2\varphi^2}-eA_0\}d\boldsymbol{p}. \tag{22}$$

Expanding it into power series of e, f and g, and picking up f^2- and g^2-terms, the result is

$$\left\{-\frac{g^2}{2\pi}p\sqrt{p^2+\mu^2}+\frac{\mu^2}{\pi}\left(\frac{g^2}{2}-\frac{f^2}{4}\right)\log\frac{2p}{\mu}\right\}_{p\to\infty}\frac{\varphi^2}{(4\pi)}. \tag{23}$$

In the case of $\varkappa=0$, this agrees perfectly $W_{\text{C.S.}}$ in (20). In the case $\varkappa\neq 0$, when we define the energy of the vacuum by (22) according to (I') and (II'), the mass term $W'_{\text{C.S.}}$ of neutral scalar meson due to the charged scalar field becomes finite by means of subtracting (23) from (22):

$$W'_{\text{C.S.}}=(f.t.)\frac{\varphi^2}{(4\pi)}, \tag{24}$$

$$W'_{\text{C.S.}}(\varkappa=0)=0. \tag{24'}$$

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* Prof. Sakata pointed out that we may possibly select (g. n. g.)-terms which appear in the ordinary calculation as those which are independent of the mass of the real (not vacuum) field at rest.

** Recently, Y. Katayama,[27] J. Rayski[11] and P. T. Matthew's[28] independently obtained the self-energy of a neutral scalar meson of which order of divergence is reduced to the logarithmic by means of suitable definition of ambiguous calculation. But this self-energy contains still (g. n. g.) term.

(b) *The self-energies of charged particles (mass μ)*

(i) *The self-energy of charged scalar particle U due to the neutral scalar φ field*

Adopting (19) as the interaction, the equation of motion for neutral scalar field φ is

$$\{\Box-\varkappa^2+2g^2U^*U\}\varphi=f\mu U^*U. \tag{25}$$

The self-energy W_s of charged scalar particle U due to the interaction (19) is, according to O. Hara,[23] as follows:

$$W_s=-\frac{1}{4\pi}\left\{g^2l^2+(\mu^2f^2-\varkappa^2g^2)\log\frac{2l}{\varkappa}+\text{finite term}\right\}_{l\to\infty}\frac{U^*U}{(2\pi)}. \tag{26}$$

According to (I′) and (II′), however, one must take into account of the alteration of the energy of the vacuum due to the deviation of equation of motion (25) from free one. In Eq. (25), the term $f\mu U^*U$ is added to equation for free case, besides the replacement $\varkappa^2\to(\varkappa^2-2g^2U^*U)$. Since this term does not involve φ, it cannot be treated as the change of the factor of φ in the equation of motion of φ. It is obscure how to introduce this term into this correspondence theoretical argument. For a moment, we will, in following, adopt the new postulate as follows.

(III′?) **Terms that cannot be regarded as the change of the factor of vacuum field quantity in its equation of motion are left out of consideration.**

and we will postpone the investigation of such a term in future. Then, taking into account of the presence of the term $2g^2U^*U$ in (25), one must, according to (I′) and (II′), define the energy of the vacuum of neutral scalar field φ as follows:

$$\frac{1}{2(2\pi)^3}\int\sqrt{l^2+\varkappa^2-2g^2U^*U}\,dl. \tag{27}$$

Expanding it into power series of g, and taking out the g^2-term, we obtain,

$$-\frac{1}{4\pi}\left\{g^2l^2-\varkappa^2g^2\log\frac{2l}{\varkappa}+\frac{1}{2}g^2\varkappa^2\right\}_{l\to\infty}\frac{U^*U}{(2\pi)}. \tag{28}$$

Using the postulate (I′), (II′) and (III′?), the self-energy W_s of charged scalar particle due to the neutral scalar field has only the logarithmic divergence:

$$W_s=\left\{-\frac{l^2f^2}{4\pi}\log\frac{2l}{\varkappa}+\text{finite term}\right\}_{l\to\infty}\frac{U^*U}{(2\pi)}. \tag{29}$$

(ii) *The electromagnetic self-energy of a charged scalar particle U*

The equation of motion of the vector potential A_μ of the electromagnetic field interacting with the charged scalar particle is:

$$\left.\begin{aligned}(\Box-2e^2U^*U)\boldsymbol{A}&=ie(U^*\operatorname{grad}U-U\operatorname{grad}U^*),\\ \Box A_4&=4\pi ic(U^*\Pi-\Pi^*U),\\ \Pi&=\frac{1}{4\pi}\left(\frac{\partial U^*}{\partial t}-eiA_4U^*\right),\end{aligned}\right\}\tag{30}$$

where Π is the canonical conjugate quantity to U. According to O. Hara, the electromagnetic self-energy of charged scalar particle is

$$W_{\text{s.e.}}=\frac{3e^2}{4\pi}\left\{l^2+\mu^2\log\left(\frac{l}{\mu}\right)+\text{finite term}\right\}_{l\to\infty}\frac{U^*U}{(2\pi)}.\tag{31}$$

From (30) we see that the alteration of the equation of motion due to this interaction appears in the three components of transverse and longitudinal waves and no change in the time component. Therefore, the energy of vacuum is

$$\frac{1}{2}\frac{1}{(2\pi)^3}\int\{3\sqrt{l^2+2e^2U^*U}-l\}d\boldsymbol{l}\tag{32}$$

from the postulates (I′), (II′) and (III′?). In the case of ordinary zero-point energy, the contributions from the longitudinal part and from the time component cancel out each other. But in Eq. (32), this is not the case because only the oscillation energy of longitudinal wave changes by the interaction with the charged particle. Expanding (32) into power series of e and taking out the e^2-term,

$$\left\{\frac{3e^2}{4\pi}l^2\frac{(U^*U)}{(2\pi)}\right\}_{l\to\infty}.\tag{33}$$

Accoringly, the electromagnetic self-energy $W'_{\text{s.e.}}$ of charged scalar meson based on (I′), (II′) and (III′?) is by subtracting (33) from (31) as follows:

$$W'_{\text{s.e.}}=\frac{3e^2\mu^2}{4\pi}\left(\log\frac{l}{\mu}+\text{finite term}\right)_{l\to\infty}\frac{U^*U}{(2\pi)}.\tag{34}$$

Formerly, Weisskopf[24] showed that the divergence of the electromagnetic self-energy of the charged spinor particle could be reduced from the quadratic to the logarithmic by means of the subtraction method based on Pauli's exclusion principle. It is very interesting that the divergence of self-energies become logarithmic not only for the spinor particle but also for the scalar particles by means of our subtraction method based on the postulates (I′), (II′) and (III′?). The first order interactions for e and f, which are left out of consideration in the definition of vacuum according to (III′?), contribute only the logarithmic divergences in (26) and (31). Therefore, even if the postulate (III′?) should be altered in future, it would be possible to conclude that the quadratically divergent terms in the self-energies of above scalar mesons disappear. It is also noteworthy that g-interactions does not contribute to divergent terms in (27)

(iii) *The self-energy of a charged spinor particle due to the neutral scalar field*

The alteration of the equation of motion of φ due to the interaction (12) is

$$\{\Box-\varkappa^2\}\varphi=f\psi^*\beta\psi. \tag{35}$$

While, the self-energy W_e of a charged spinor particle due to the interaction (12) is[23)25)]

$$W_e=-\frac{3\mu f^2}{4\pi}\left\{\log\frac{2l}{\mu}+\text{finite term}\right\}_{l\to\infty}\psi^*\beta\psi. \tag{36}$$

According to (III'?), (35) does not give rise to the change of zero-point energy so that, (36) is again the self-energy of a charged spinor particle due to the neutral scalar field in the argument based on (I'), (II') and (III'?),

(iv) *The electromagnetic self-energy of a charged spinor particle*

The equation of motion of the electromagnetic potential A_μ interacting with the charged spinor field is

$$\Box A_\mu=e\psi^*\alpha_\mu\psi.$$

According to (III'?), the electromagnetic self-energy $W_{e.e.}$ of a charged spinor particle may be the same one as in usual calculation ;[38)] i.e.

$$W_{e.e.}=\frac{3\mu e^2}{2\pi}\left\{\log\frac{2l}{\mu}+\text{finite term}\right\}_{l\to\infty}\psi^*\beta\psi. \tag{38}$$

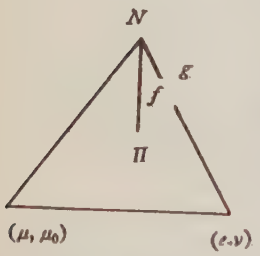

Fig. 1

The results for the self-energies of a charged spinor particle remain unchanged in the case of using the postulates (I'), (II') and (III'?). But these features would, of course, depend on the validity of postulate (III'?).

§ 5. Transitions of elementary particles through the vacuum particles

We have shown in § 3 that the postulates (I') and (II') has influence upon the process of γ-decay of neutral meson. This fact means more generally that (I') and (II') would play an important role for the transitions of elementary particles through the intermediation of the vacuum particles. In all problems which connect with the vacuum particles, the definition of vacuum always comes into question. We take π-μ decay as the simplest example of such problems. L. I. Schiff[40)] investigated the case of Fig. 1 in which π-mesons, nucleons, μ-mesons and light particles interact each other. This is the important problem for determining the model of two mesons, π and μ. Since the π-μ decay or π-e decay based on the model in Fig. 1 is intermediated through the vacuum nucleon pair, the divergence occurs and Schiff's conclusion intimately depends on the way of cut off. In fact, Steinberger was led to the

different conclusion from Schiff by using Pauli's regularization method. Such a divergence depends on the physical character of the vacuum and on the structure of elementary particles. We analyse it from the view point of (I′) and (II′). Accoding to Schiff, we take π-meson φ (mass $\varkappa$) as scalar, and μ meson ϕ_μ. neutral meson ϕ_ν and nucleon ψ (mass μ) as spinor. By means of following interactions:

π-meson-nucleon. $f\psi^*\tau\beta\psi\varphi$ (scalar coupling), (28)

nucleon-μ-neutral meson: $g(\psi^*\tau\beta\psi)(\phi_\nu{}^*\beta\phi_\mu)$ (scalar coupling), (39)

the matrix element $H_{\pi\to\mu}$ of the $\pi\to\mu+\nu$ decay through nucleon pairs, is

$$H_{\pi\to\mu}=-\frac{fg}{\pi^2}\left\{\int_0^\infty\frac{p^4dp}{(p^2+\mu^2)^{3/2}}+\varkappa^2\int\frac{p^4dp}{(p^2+\mu^2)^{3/2}}\frac{1}{(4p^2+4\mu^2-\varkappa^2)}\right\}(\tau\tau)\phi_\nu\beta\phi_\mu\varphi \tag{40}$$

by using ordinary perturbation calculation. i.e. it has the quadratic and logarithmic divergences. On the other hand, the equation of motion for nucleon is

$$\{(a,\boldsymbol{p})+\beta(\mu+\tau f\varphi)+\tau gL\}\psi=0, \tag{41}$$

$$L=(\phi_\nu{}^*\beta\phi_\mu). \tag{41′}$$

Accordingly, the energy of vacuum nucleon is, according to (I′) and (II′)

$$-\frac{2}{(2\pi)^3}\int d\boldsymbol{p}\sqrt{p^2+(\mu+\tau f\varphi+\tau gL)^2}\,. \tag{42}$$

Expanding the above expression into power series of g and f, the fg-term is as follows:

$$-\frac{fg}{\pi^2}\int\frac{p^4dp}{(p^2+\mu^2)^{3/2}}(\tau\tau)\phi_\nu{}^*\beta\phi_\mu\varphi. \tag{43}$$

Therefore, according to (I′) and (II′), by subtracting (43) from (40), the matrix element $H'_{\pi\to\mu}$ of $\pi\to\mu$ decay becomes

$$H'_{\pi\to\mu}=-\frac{fg\varkappa^2}{\pi^2}\left\{\int\frac{p^4dp}{(p^2+\mu^2)^{3/2}}\cdot\frac{1}{(p^2+4\mu^2-\varkappa^2)}\right\}(\tau\tau)\phi_\nu{}^*\beta\phi_\mu\varphi, \tag{44}$$

i.e. which is of the logarithmic divergence, and

$$H'_{\pi\to\mu}(\varkappa=0)=0. \tag{44′}$$

§ 6. *C*-meson theory

There is Sakata's[6)23)25)] *C*-meson theory as the substantialistic method extricating from the diverging self-energy of charged particles. Since, according to above argument, the self-energies of elementary particles are, however, altered by the definition of vacuum, *C*-meson theory must also be modified slightly. In this case, it is sufficient to regard the neutral scalar field in § 4 as *C*-meson.

Since the self-energy of an electron due to the C-meson and the electromagnetic self-energy of an electron are not altered by using our postulates, the self-energy of an electron are not altered by using our postulates, the self-energy of an electron becomes finite by setting a well known relation[1,3]

$$f^2=2e^2 \tag{45}$$

from (25) and (38).

The condition for cancelling the diverging part of the electromagnetic self-energy of charged scalar meson by means of that due to C-meson becomes very simple, because their quadratic divergence in usual calculation was reduced to the logarithmic one by using (I′) and (II′), that is, from (29) and (34), the condition for obtaining the finite self-energy of charged scalar particle is

$$f^2=3e^2. \tag{46}$$

g^2 in interaction (19) may be taken to be arbitrary, and so the most simple case is $g=0$.

The problem of the self-energy of C-meson is of less difficulty than usual. Its self-energy (24) due to the charged scalar meson is finite, and its self-energy due to charged spinor particle is of the logarithmic divergence. (cf. Eq. (18)). But the self-energy of C-meson cannot be finite by means of the mixture of only the charged scalar mesons and the charged spinor particles. However, (1) the vector mesons are now left out of consideration. (2) The nature of C-meson is ambiguous in many points and it is obscure with how many particles the C-meson interacts. From these reasons we cannot unambiguously conclude about the self-energy of C-meson. These divergencies which still remain after using postulate (I′) and (II′) seems to connect with the problem of structure of elementary particles, because they depend on the mass of real field (not on that of vacuum field).

§ 7. Concluding remarks and summary

Determining the energy of vacuum by using the postulates (I′) and (II′) (and (III′?)), usual non-gauge-invariant results appearing in various problems can be made gauge-invariant. The internal conflict, that the non-gauge-invariant results have ever been obtained in the theory in which the gauge-invariance had been verified formally, seems to consist in the incompleteness of usual definition of vacuum. The postulates (I′), (II′) and (III′?) have an important effect not only upon the problem of gauge-invariancy, but also upon all processes connected with vacuum particles. By means of this method, divergencies of self-energies of elementary particle can be reduced to lower order. Especially, in the same manner as Weisskopf's subtraction of vacuum, by which the order of divergence of self-energy of charged spinor particle could be reduced to the logarithmic

order, the divergence of self-energy of scalar mesons is also reduced to the logarithmic order by means of the subtraction method based on the postulates (I′), and (II′).

Moreover, these postulates reduce the order of divergence for the various divergencies appearing in usual calculation of transition between mesons through vacuum particle pairs.

Hitherto, it is known that the direct interaction between spinor fields make divergencies of higher order. This problem must be investigated from the point of view based on above arguments of the vacuum.

The definition of vacuum based on (I′) and (II′) is, however, of only the correspondence theoretical character and it is the purpose of this paper to point out that to fomulate such a correspendence theoretical argument into complete quantum field theoretical formalism will be one of the important works to be accomplished.

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High Energy Nuclear Evaporation

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Introduction

We need the detailed knowledges on evaporation process for the analysis of various experiments on nuclear stars, which have been rapidly increasing in last several years. Previously, we studied general characters of nuclear stars, but the evaporation process was discussed too little there[1].

Main defects of our previous treatment are as follows: i) neglect of α-particle emission, ii) rough estimation of fluctuation. We improve these points and extend our calculation in this paper.

The problem to be considered here is "How many and what kind of particles are evaporated from the certain mother nucleus with the given excitation energy?" Applications of our results to various high energy nuclear phenomena will be done in a separate paper.

1. Preliminary remarks

Adopting the degenerate Fermi gas model for a nucleus[2], we have the following relations between the entropy S, the excitation energy X, and the temperature T for the nucleus (N, Z), i.e., the nucleus of N neutrons and Z protons $(N+Z=A=\text{mass number})$.

$$X\approx\left\{1-\frac{1}{9}\left(\frac{N-Z}{N+Z}\right)^2\right\}\frac{\pi^2}{4E_F}AT^2,$$

$$S(X)\approx\left\{1-\frac{1}{18}\left(\frac{N-Z}{N+Z}\right)^2\right\}\left(\frac{AX}{E_F}\right)^{\frac{1}{2}},$$

E_F=Fermi energy.

In these expressions, each second term in braces is the small correction depending on $N-Z$. We may, therefore, safely neglect these correction terms. (However we can not neglect the contribution from the term depending on $N-Z$ for the binding energy of a nucleon; see below and ref. 1). Inserting Serber's value 22 MeV for Fermi energy E_F[3], they become

$$\left.\begin{aligned} &X\approx\frac{A}{10}T^2, \\ &S_A(X)\approx 0.63_2(AX)^{\frac{1}{2}}, \end{aligned}\right\} \qquad (1.1)$$

X, T measured in MeV.

Here we do not take into account so-called Bardeen's remark[4], for (1.1) gives the energy distribution of evaporated protons which is in better agreement with the experiment of Perkins et al.[5]. In other words, we are inclined to regard (1.1), so to speak, as the empirical formula.

Bohr and Wheeler[6] gave the following expression for the main part of the total binding energy of a nucleus (N, Z)*;

$$B(N, Z)=AE_b-\frac{1}{2}b_A(Z-Z_A)^2, \qquad (1.2)$$

where E_b means the average binding energy of a nucleon for the *most stable* nucleus (N_A, Z_A) with mass number A. E_b and b_A are the slowly varying quantities of A, and b_A was tabulated in Bohr-Wheeler's original paper[6].

Then the binding energy E_p and E_n of a proton and a neutron for the nucleus (N, Z) are given by

$$\left.\begin{aligned} E_p&=\frac{\partial B}{\partial Z}\Big|_{N=\text{const}}\fallingdotseq E_b-b_A(Z-Z_A), \\ E_n&=\frac{\partial B}{\partial N}\Big|_{Z=\text{const}}\fallingdotseq E_b-b_A(N-N_A), \end{aligned}\right\} \qquad (1.3)$$

respectively. The differentials of the slowly varying quantities are neglected.

Next, we will consider the nucleus $(N-n, Z-p)$ $(a=n+p$, assuming $a\ll A)$, which is produced from the "stable" mother nucleus (N, Z) by carrying away n neutrons and p protons. The most probable charge Z_{A-a} for a nucleus of $(A-a)$ nucleons is approximately

$$Z_{A-a}\approx Z_A-a\frac{dZ_A}{dA}=Z_A-a\frac{1-\mu}{2},$$

where

$$\frac{d(N_A-Z_A)}{dA}=\frac{d(A-2Z_A)}{dA}=\mu.$$

It is easily calculated from the Bethe-Weizsäcker's semi-empirical formula for the binding energies[7][8]. Substituting this expression into (1.3), we obtain the following formula for E_p and E_n of the nucleus $(N-n, Z-p)$:

* (1.2) is derived from eq. (6) of Bohr-Wheeler's paper (ref. 6).

$$\left.\begin{aligned} E_p &\approx E_b - \beta\left(n - \frac{1+\mu}{1-\mu}p\right), \\ E_n &\approx E_b + \beta\left(n - \frac{1+\mu}{1-\mu}p\right), \end{aligned}\right\} \tag{1.4}$$

$$\beta = b_A \frac{1-\mu}{2}.$$

These results were used in I. The line

$$n = \frac{1+\mu}{1-\mu}p$$

represents the "Heisenberg valley" near the point (N, Z) in the nuclear chart.

The binding energy of an α-particle in a nucleus is given by

$$E_\alpha \approx 2E_p + 2E_n - (\textit{total binding energy of an } \alpha\textit{-particle})$$
$$\approx 4E_b - 28.2 \text{ MeV}.$$

For heavy nuclei ($A \approx 100$), this is

$$E_\alpha \approx 4 \text{ MeV}.$$

We may neglect the change of the Coulomb barrier height V throughout the evaporation process. It is justified by experiments on energy distribution of emitted protons[5], and by estimation of thermal expansion of a nucleus[9].

2. The evaporation theory

According to Weisskopf[10], the probability per unit time that the nucleus $(N, Z; X)$, i.e., nucleus (N, Z) with excitation energy X, *evaporates* a "j"-particle with the energy ξ, $\xi + d\xi$ is given as follows

$$W_j(X, \xi)d\xi = \frac{(\pi R^2)\cdot g_j m_j}{\pi^2\hbar^2}(\xi - V_j - E_j)\exp\left[S_{A-a_j}(X-\xi) - S_A(X)\right]d\xi. \tag{2.1}$$

We write mass, charge and statistical weight (due to its spin) of this "j"-particle as $m_j = a_j m$, $z_j e$ and g_j, respectively; where m means nucleon mass. Furthermore, we use the following notations.

R = radius of mother nucleus
$= r_0 A^{1/3}$, $r_0 = 1.37 \times 10^{-13}$ cm[3],
V_j = Coulomb barrier height for j-particle,
E_j = binding energy of j-particle,

$S_A(X)$ or $S_{A-a_j}(X)$ means the entropy of the nucleus with excitation energy X before or after the evaporation. The kinetic energy of an emitted j-particle is $\xi - E_j$.

For $X \gg \xi$, we may use the following approximation,

$$S_{A-a_j}(X-\xi)-S_A(X) \approx -\frac{a_j}{10}T-\frac{\xi}{T}. \tag{2.2}$$

$T=(dS_A(X)/dX)^{-1}$ means the temperature (measured in MeV) of this excited nucleus with excitation energy X. Then the energy distribution of an evaporated j-particle is

$$(\xi-E_j-V_j)\exp\left(-\frac{\xi-E_j-V_j}{T}\right).$$

The mean kinetic energy of an evaporated j-particle is V_j+2T, and the mean decrease in excitation energy of the mother nucleus due to emission of j-particle is

$$V_j+E_j+2T. \tag{2.3}$$

The "j-particle width" $\Gamma_j(X)$ is obtained by the integration,

$$\Gamma_j(X)=\int_{V_j+E_j}^{X} W_j(X,\xi)\,d\xi$$

$$\approx \frac{g_j a_j}{2\pi}\frac{A^{2/3}}{K}T^2\exp\left(-\frac{a_jT}{10}-\frac{V_j+E_j}{T}\right), \tag{2.4}$$

where $K=\hbar^2/2mr_0^2=11.1$ MeV. For instance, proton width $\Gamma_p(X)$, ($g_p=2$, $m_p=m$, $V_p=V$) neutron width $\Gamma_n(X)$ ($g_n=2$, $m_n=m$, $V_n=0$) and α-particle width $\Gamma_\alpha(X)$ ($g_\alpha=1$, $m_\alpha \fallingdotseq 4m$, $V_\alpha \fallingdotseq 2V$) are as follows:

$$\Gamma_p(X)\approx\frac{1}{\pi}\frac{A^{2/3}T^2}{K}\exp\left(-\frac{T}{10}-\frac{E_p'+V}{T}\right), \tag{2.5p}$$

$$\Gamma_n(X)\approx\frac{1}{\pi}\frac{A^{2/3}T^2}{K}\exp\left(-\frac{T}{10}-\frac{E_n}{T}\right) \tag{2.5n}$$

and

$$\Gamma_\alpha(X)\approx\frac{1}{\pi}\frac{A^{2/3}T^2}{K}\exp\left(-\frac{4T}{10}-\frac{E_\alpha+2V}{T}\right). \tag{2.5a}$$

For later use, we write the ratio $\Gamma_\alpha(X)/\Gamma_p(X)$ as,

$$\frac{\Gamma_\alpha(X)}{\Gamma_p(X)}\approx k\exp\left(-\frac{E_\alpha+V-E_p}{T}\right), \quad k=2e^{-\frac{3T}{10}}.$$

The approximate expression of the fission width is[6)13)]

$$\Gamma_f(X)\approx\frac{1}{2\pi}T\exp\left(-\frac{E_f}{T}\right), \tag{2.5f}$$

where E_f means the fission threshold, and we assume that the small deformation

of the mother nucleus causes the fission process (cf. ref. 6).

Formerly Tomonaga[11)12)] calculated the coefficient of heat conduction and of viscosity of nuclear matter and showed that they were very large. Comparison of the heat conduction with the particle evaporation leads us to the next two conclusions:

(i) It is very rare that a part of a nucleus is locally superheated and evaporates any particle.

(ii) After evaporating a particle, the residual nucleus will reach the thermal equilibrium state before next evaporation takes place.

The former (i) was previously used to conclude the possibility of fission process in very large cosmic ray stars[13)]. We see from (ii) that a particle is always emitted from a nucleus in thermal equilibrium.

3. The diffusion equation for evaporation process

Let us suppose the mother nucleus $(N_0, Z_0; X_0)$ at the biginning (time $t=0$). Various particles will be evaporated from this nucleus. The probability $F(N, Z, X, t)dX$ of finding a nucleus (N, Z) with excitation energy between X, $X+dX$ at time t satisfies the following simultaneous diffusion-type equations:

$$\frac{\partial F(N, Z, X, t)}{\partial t} = \sum_j \int_{X+V_j+E_j}^{\infty} F(N+a_j-z_j,\ Z+z_j,\ X',\ t) \times W(X', X'-X)dX' - \sum_j F(N, Z, X, t) \int_{V_j+E_j}^{X} W_j(X, \xi)d\xi. \tag{3.1}$$

The initial condition is, of course, as follows,

$$F(N, Z, X, t) = \delta_{NN_0}\delta_{ZZ_0}\delta(X-X_0). \tag{3.2}$$

Solving (3.1) under the initial condition (3.2), we get the desired answer to our problem of nuclear evaporation: i.e.,

$$F(N, Z) \equiv \int_0^{min\,(V_j+E_j)} F(N, Z, X, +\infty)dX. \tag{3.3}$$

This is the yield probability of a residual nucleus (N, Z) from the mother nucleus $(N_0, Z_0; X_0)$. Other required quantities are derived from $F(N, Z, X, t)$.

It may, however, be necessary to perform very long numerical calculations to solve our problem in this way, although this was done by Horning and Baumhoff for some simplified case[14)]. (Also interesting formal considerations of eq. (3.1) were done by Baumhoff.[15)]) Avoiding this direct attack to our problem, we prefer to examine in the first place the average behaviour of evaporating nuclei, and then take into account the effect of fluctuations.

4. Mean behaviour of evaporation process

Now we discuss the mean behaviour of the evaparating nuclei. Mean energy $E^{(j)}$ carried out of the nucleus $(N, Z; X)$ by emission of j-particle is given by

$$E^{(j)}=V_j+E_j+2T,$$
$$T=3.17(X/A)^{\frac{1}{2}}. \tag{4.1}$$

The nucleus (N, Z) is the intermediate one in the evaporation process, after emission of n neutrons and p protons, i.e.

$$N=N_0-n, \quad Z=Z_0-p,$$
$$(N+Z=A, \quad N_0+Z_0=A_0, \quad n+p=a)$$

and we assume $$Z_0=Z_{A_0}.$$

When the nucleus evaporates, on the average, one nucleon, the mean decrease of its excitation energy is equal to

$$-\frac{\partial X}{\partial a}=\frac{\sum_j \Gamma_j \cdot E^{(j)}}{\sum_j \Gamma_j a_j} \tag{4.2}$$

While the probability or the width for emitting j-particle is

$$\Gamma_j(N, Z; X)=\frac{g_j a_j}{2\pi}\frac{A^{2/3}T^2}{K}\exp\left(-\frac{a_j}{10}T-\frac{V_j+E_j}{T}\right). \tag{4.3}$$

This set of equations (4.2) and (4.3) determines the mean number of j-particles ν emitted before the nucleus is cooled down to the excitation energy X.

$$\nu_j(X_0, X)=\int_X^{X_0}\frac{\Gamma_j dX}{\sum_k \Gamma_k E^{(k)}} \tag{4.4}$$

Thus the average number $\langle \nu_j(X_0) \rangle$ of j-particles evaporated throughout the whole process is equal to $\nu_j(X_0, \Gamma_j+E_j)$. Of course, number of neutrons and protons n and p carried away from the mother nucleus are given by

$$p=\sum z_j \nu_j, \quad n=\sum (a_j-z_j)\nu_j, \quad (j=n, p, \alpha, \cdots\cdots).$$

The curve $\nu_j=\nu_j(X_0, X)$ $(j=n, p, \alpha, \cdots\cdots)$ in ν_j-space represents the mean behaviour of the evaporation process and we call this "evaporation path". This is a generalization of Bagge's mean cooling law.[16]

For practical purpose, it is sufficient only to take into account neutrons, p otons and α-particles, since other particles have much smaller widths.

(A) Case (np) Consideration of only neutron and proton emission (case (np)) was discussed briefly in I. Let us examine this simplest case more closely. In this case, we must solve the following set of equations (of course $\nu_n=p$ and $\nu_n=n$)

$$\left.\begin{aligned} -\frac{\partial X}{\partial(n+p)} &= \frac{\Gamma_n(E_n+2T)+\Gamma_p(V+E_p+2T)}{\Gamma_n+\Gamma_p}, \\ \frac{dn}{dp} &= \frac{\Gamma_n}{\Gamma_p} \quad n=p=0 \quad \text{for} \quad X=X_0. \end{aligned}\right\} \tag{4.5}$$

Explicit expression for each quantity is as follows.

$$E_n \approx E_b+\beta\left(n-\frac{1+\mu}{1-\mu}p\right),$$

$$E_p \approx E_b-\beta\left(n-\frac{1+\mu}{1-\mu}p\right),$$

$$\frac{\Gamma_n}{\Gamma_p} \approx \exp\left(\frac{V+E_p-E_n}{T}\right) \approx \exp\left\{\frac{V+2\beta\left(n-\frac{1+\mu}{1-\mu}p\right)}{T}\right\}.$$

$$T=3.17\left(\frac{X}{A_0-n-p}\right)^{\frac{1}{2}}, \tag{4.6}$$

For $A_0=100$, they are

$$\left.\begin{aligned} &E_b \sim 8 \text{ Mev}, \qquad V \sim 6 \text{ Mev}, \\ &\beta \sim 0.9, \quad \frac{1+\mu}{1-\mu} \sim 1.7 \end{aligned}\right\} \tag{4.7}$$

Various properties of the solution of these simultaneous equations can be obtained from consideration on following simplified models.

(i) Evaporation process with constant temperature.

This neglection of cooling allows us to treat the only one equation

$$\frac{dn}{dp}=\exp\left\{\frac{V+2\beta\left(n-\frac{1+\mu}{1-\mu}p\right)}{T}\right\}. \tag{4.8}$$

Its solution (under the condition $n=0$ for $p=0$) is,

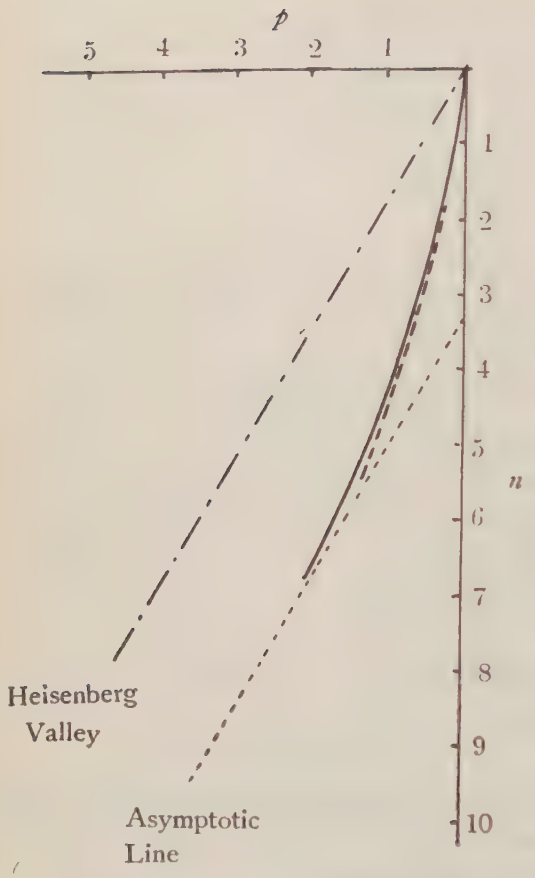
$$\frac{1+\mu}{1-\mu}\,\frac{e^{\frac{2\beta n}{T}}-1}{e^{2\beta\frac{1+\mu}{1-\mu}p\,\frac{1}{T}}-1}=e^{\frac{V}{T}}. \tag{4.9}$$

For large n and p, this tends

$$2\beta\left(n-\frac{1+\mu}{1-\mu}p\right)-V=0, \tag{4.10}$$

which is parallel to the "Heisenberg valley". (See Fig. 1.)

(ii) We assume that the ratio Γ_n/Γ_p, E_p and E_n are constant through



—— curve (4.9) with (4.7) (for $T=3$ Mev)
----- solution of (4.5), (4.6) and (4.7)
($A_0=100$, $X_0=100$ Mev)

Fig. 1.

evaporation process.* In this case it needs only to solve the equation

$$-\frac{\partial X}{\partial a}=X_b+2T,$$

$$X_b=E_b+\frac{\Gamma_p}{\Gamma_n+\Gamma_p}V, \quad n+p=a, \tag{4.11}$$

$$E_p=E_n=E_b, \quad T=3.17\left(\frac{X}{A-a}\right)^{\frac{1}{2}}.$$

The initial condition is $a=0$ for $X=X_0$. The solution can easily be found as

$$\frac{A_0-a}{A_0}=\frac{X_b+2T_0-\frac{1}{a}T_0^2}{X_b+2T_a-\frac{1}{a}T_a^2}$$

$$\times\left[\frac{\sqrt{\ }+a-T_0}{\sqrt{\ }-a+T_0}\,\frac{\sqrt{\ }-a+T_a}{\sqrt{\ }+a-T_a}\right]^{\frac{a}{\sqrt{\ }}} \tag{4.12}$$

where

$$\left.\begin{aligned}&\sqrt{\ }\equiv\sqrt{a(a+X_b)}\\&X=\frac{A_0-a}{a}T^2, \quad a\approx 10\end{aligned}\right\}$$

T_0 denotes the initial temperature of the nucleus and T_a the temperature of the nucleus just after evaporating a nucleons. This solution (4.12) was also obtained by Perkins et al.[5] It is worth while to note that a/A_0 depends not on A_0 but only on T_a and T_0. We think that the evaporation process stops when the excitation energy decreases down to $X=X_b=E_b+(\Gamma_p V/\Gamma_n+\Gamma_p)$. (See Fig. 2.)

We see from the above models that the evaporation path determined by the exact equation (4.5) has its end point close the asymptotic line (4.10) and its path length is nearly equal to that of (4.12). The results of numerical integration of (4.5) are given in Table 1 and in Fig. 3.

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* Y. Fujimoto and Y. Yamaguchi, Lecture at the Cosmic-Ray Symposium (Nagoya University; Dec. 12, 1948).

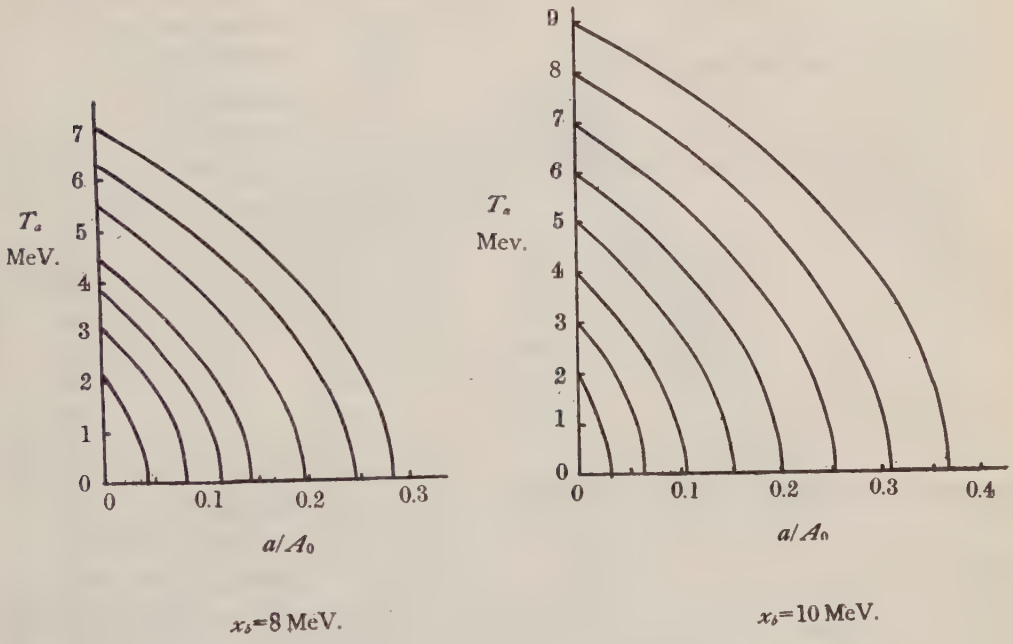

Fig. 2. Curves of (4.12)

(B) Case (npa). Next we take into account the α-particle emission (case (npa)).* Then the cooling law becomes,

$$-\frac{\partial X}{\partial a}=\frac{\Gamma_n(E_n+2T)+\Gamma_p(E_p+V+2T)+\Gamma_\alpha(E_\alpha+2V+2T)}{\Gamma_n+\Gamma_p+4\Gamma_\alpha},$$

$$\frac{dn}{dp}=\frac{\Gamma_n+2\Gamma_\alpha}{\Gamma_p+2\Gamma_\alpha}, \tag{4.13}$$

$$a=n+p, \qquad n=\nu_n+2\nu_\alpha, \qquad p=\nu_p+2\nu_\alpha.$$

If we adopt the expression (2.5a) for α-particle width, the number of evaporated α-particles becomes much smaller than that observed in cosmic-ray stars (Table 2). This is readily seen as follows: firstly,

$$\frac{\nu_\alpha}{\nu_p}\approx\left\langle\frac{\Gamma_\alpha}{\Gamma_p}\right\rangle_{AV}.$$

and

$$\frac{\Gamma_\alpha}{\Gamma_p}\approx 2e^{-\frac{3T}{10}}\cdot e^{-\frac{E_\alpha+V-E_\alpha}{T}}. \tag{4.14}$$

* In the preliminary report,[17] very crude approximation:

$$\frac{dn}{dp}=\frac{\Gamma_n}{\Gamma_p}\left(1-2\frac{\Gamma_\alpha}{\Gamma_p}\right).$$

was used and the results stated there are considerably different from the ones of this paper.

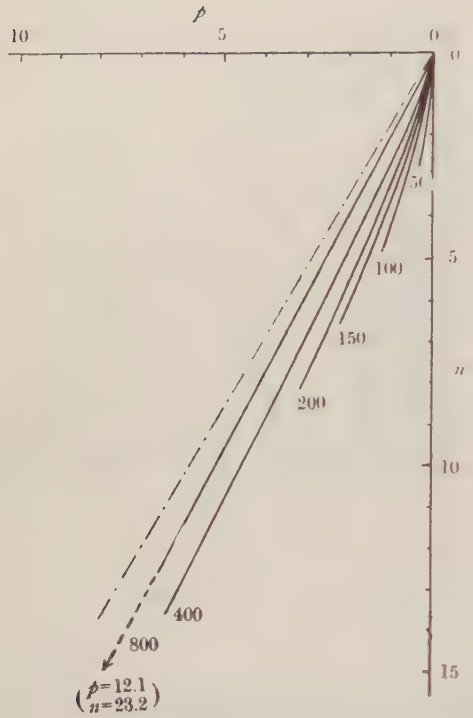

Fig. 3. Solution of (4.5), (4.6), (4.7)

This expression has its maximum at

$$T\sim\sqrt{\frac{10}{3}(E_a+V-E_p)}\ .$$

Thus, for the numerical values

$$\left.\begin{aligned} &E_a\sim 4\text{ Mev},\ E_p\sim 8\text{ Mev},\ V\sim 6\text{ Mev},\\ &(A\approx 100),\end{aligned}\right\}\qquad (4.14)$$

this is

$$\left(\frac{\Gamma_a}{\Gamma_p}\right)_{max}\sim 2e^{-2}=0.27$$

While, the observed ratio ν_a/ν_p in cosmic ray stars is somewhat larger, i.e.$\sim$0.5.[18]

This discrepancy may be possibly due to wrong A-dependence of assumed entropy of a nucleus. It is our regret that we can not estimate the above possible effect. Tentatively, we assume three constant values, 1/2, 1, and 2 for k (4.14) and solve the equations numerically. ν_a/ν_p ratio observed in cosmic ray stars

Table I. Numbers of evaporated particles; case (n_p) (obtained from (4.5))

(i) $\begin{cases} A_0=100 \\ E_b=8\text{ Mev},\ V=6\text{ Mev} \\ \beta=0.9,\quad \dfrac{1+\mu}{1-\mu}=1.7 \end{cases}$

X_0	$<p>$	$<n>$
50 Mev	0.27	2.72
100	1.19	4.84
150	2.20	6.54
200	3.15	8.15
400	6.4	13.6
800	12.1	23.2

(ii) $\begin{cases} A_0=100 \\ E_b=8\text{ Mev},\ V=6\text{ Mev} \\ \beta=0.0,\quad \dfrac{1+\mu}{1-\mu}=1.7 \end{cases}$

X_0	$<p>$	$<n>$
100 Mev	0.48	6.52

(iii) $\begin{cases} A_0=200 \\ E_b=8\text{ Mev},\ V=9\text{ Mev} \\ \beta=0.5,\quad \dfrac{1+\mu}{1-\mu}=2.0 \end{cases}$

X_0	$<p>$	$<n>$
200 Mev	1.53	10.82

Table II. Numbers of evaporated particles; case (n/α), (solutions of (4.13), (2.5 a), (4.14))

$A_0=100$

X_0	$<\nu_n>$	$<\nu_p>$	$<\nu_\alpha>$
200 Mev	7.9	2.6	0.67
400	18.5	5.0	1.3
800	22.0	9.4	1.9

Table III. Numbers of evaporated particles, case $(n\gamma\alpha)$.

$$\left.\begin{matrix} A_0=100 \\ E_b=8\ \text{Mev} \\ E_\alpha=4\ \text{Mev} \\ V=6\ \text{Mev} \end{matrix}\right| \quad \begin{matrix} \frac{1+\mu}{1-\mu}=1.7 \\ \beta=0.9 \end{matrix} \qquad \frac{\Gamma_\alpha}{\Gamma_p}=k\exp\left(\frac{E_p-E_\alpha-V}{T}\right)$$

X_0	$k=2$			$k=1$			$k=1/2$		
	$\langle\nu_n\rangle$	$\langle\nu_p\rangle$	$\langle\nu_\alpha\rangle$	$\langle\nu_n\rangle$	$\langle\nu_p\rangle$	$\langle\nu_\alpha\rangle$	$\langle\nu_n\rangle$	$\langle\nu_p\rangle$	$\langle\nu_\alpha\rangle$
100 Mev	4.7	0.8_6	0.4_8	4.7	1.0_0	0.2_7			
200	7.7	1.7_8	1.5_8	7.8	2.3_1	0.9_1	7.8_8	2.6_4	0.4_8
400				13.0	4.1_7	2.4_4			
800				20.8	6.7_2	5.4^6			

seems to be most favorable to the value ~ 1 for k. (See Table 3.)

(C) It is worth while to point out that the value $\langle\nu_p(X_0)\rangle+\langle\nu_\alpha(X_0)\rangle$, i.e., average number $\langle P(X_0)\rangle$ of prongs in one star, for given X_0 (and also A_0) is not sensitive to assumed values of k (Table 3). Thus we may hope that the average number of prongs (for given X_0) will not much change even if k is a rather complicated function of T.

Therefore it may be allowed to divide the value $\langle P(X_0)\rangle$ thus obtained into two parts, $\langle\nu_p(X)\rangle$ and $\langle\nu_\alpha(X)\rangle$, such that their ratio $\langle\nu_\alpha\rangle/\langle\nu_p\rangle$ is in agreement with the experimental one. We plot in Fig. 4 the values $\langle\nu_p\rangle$ and $\langle\nu_\alpha\rangle$ *versus* the initial excitation energy X_0 (for $A_0=100$) determined in this way.

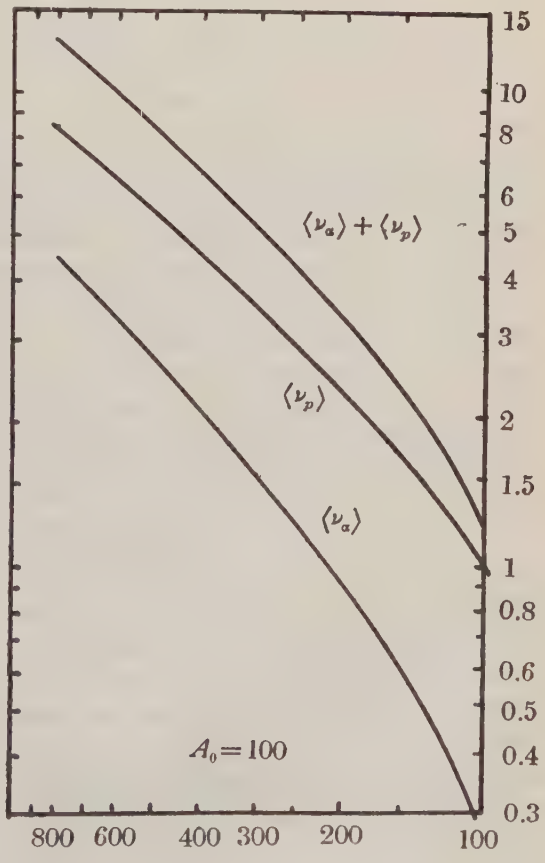

Fig. 4

5. Fluctuation

As Weisskopf's distribution has a broad maximum, actual kinetic energies of evaporated particles are not always equal to their mean values, so that the number of eva-

porated particles show some probability distributions. We discuss the effect of such fluctuations

(A) Fluctuation of a (number of nucleons carried away from the mother nucleus). We may regard the evaporation of nucleons and of α-particles as independent to each other, for the evaporation of an α-particle does not result in change of binding energies. Therefore we treat them separately.

We discussed in I the fluctuation of number of evaporated nucleons $\nu' = \nu_p + \nu_n$. The probability $f(\nu')$ for evaporating just a nucleon is well approximated by Gaussian function as follows:

$$f(\nu') = \frac{1}{\sqrt{\pi}\,\Delta\nu'} \exp\left[-\frac{(\nu' - \langle\nu'\rangle)^2}{2(\Delta\nu')^2}\right], \tag{5.1}$$

where

$$\Delta\nu' = \frac{\sqrt{2}\,T}{E_b}\sqrt{\langle\nu'\rangle}\,.$$

The probability $g(\nu_\alpha)$ for evaporating ν_α α-particles will be Poisson's type,

$$g(\nu_\alpha) = e^{-\langle\nu_\alpha\rangle} \cdot \frac{\langle\nu_\alpha\rangle^{\nu_\alpha}}{\nu_\alpha!},$$

since the evaporation of α-particles will have little correlations.

We get the distribution of total number $\nu = \nu' + \nu_\alpha$ of evaporated particles by combining these two distributions. Comparing $\Delta\nu'$ with $\Delta\nu_\alpha = \langle\nu_\alpha\rangle'$, we see that $\Delta\nu' > \Delta\nu_\alpha$ in most cases. Thus this distribution is mainly governed by (5.4).

On the contrary, the distribution of total number $a = \nu' + 4\nu_\alpha$ of nucleons carried away is mainly determined by that of α-particles, since $\Delta\nu' < 4\Delta\nu_\alpha$ in most cases. For some special cases, such as low energy evaporation process, $4\Delta\nu_\alpha$ becomes smaller than $\Delta\nu'$, and the distribution of ν' plays an important role.

(B) Fluctuation on n or p (Case (np)). Formerly we assumed binary distribution for the number p or n of evaporated protons or neutrons, i.e.

$$\frac{(n+p)!}{n!\,p!}\,\frac{\langle n\rangle^n\langle p\rangle^p}{(\langle n\rangle + \langle p\rangle)^{n+p}}. \tag{5.2}$$

This neglection of correlation in successive evaporation of a nucleon is only valid for the case of small $\langle p\rangle$ and $\langle n\rangle$.

When $\langle p\rangle$ and $\langle n\rangle$ becomes considerably larger, i.e. larger than the breadth of Heisenberg valley, we can not neglect the effect of change of binding energies, which imposes serious correlation between them. In this case, we may treat their fluctuation as Markoff's chain.

For simplicity we put $(1+\mu)/(1-\mu) = 1$, then the ratio of the widths is

$$\frac{\Gamma_n}{\Gamma_p} = \exp\left\{\frac{V - 2\beta(n-p)}{T}\right\} \tag{5.3}$$

Assuming the constant temperature T, we get the diffusion equation for the

probability $F(n, p)$ for finding the nucleus $(N-n, Z-p)$. We get its steady solution $F(n, p)$, which depends only on $n-p$, as follows:

$$F(n, p)=C[1+e^{\frac{2\beta(n-p)-V}{T}}]e^{-\frac{\beta}{T}\left(n-p-\frac{V}{2\beta}\right)\left(n-p-\frac{V}{2\beta}+1\right)}, \tag{5.4}$$

where C is determined by the normalization condition,

$$\sum_{n-p=-\infty}^{+\infty} F(n, p)=1.$$

Of course, we should use the average temperature T suitably chosen.

6. Summary and acknowledgement

We calculated the prong number distribution and its relation to excitation energy on the basis of Weisskopf's evaporation theory. These results are applicable to high energy nuclear reactions, meson stars, cosmic ray stars, etc.

The authors express their hearty thanks to members of Tokyo University and St. Paul University for their kind discussions on this work.

It seems to be necessary some additional remark to avoid misunderstanding. We can not give the complete explanation of nuclear stars only using the evaporation theory considered here. But we consider that particles emitting in the star are roughly classified in two classes: the one is so-called "Heisenberg nucleons" and the other is evaporated particles. This classification is justified by the recent experiments.* Thus we considered the *ideal* evaporation processes somewhat in details in this paper. To make a discussion of nuclear stars more complete, we must consider the Heisenberg nucleons etc.

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* Cf. ref. 5), also see C. F. Powell et al, Phil. Mag. **40** (1949), 862.

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Production of π-Mesons in Nucleon-Nucleon Collisions near the Threshold Energy

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§ 1. Introduction

In connection with recent experiments of artificial meson production by nucleon-nucleon collisions, several authors[1),2),3)] have calculated the cross-sections for these phenomena. However, their results are more or less unreliable because of the arbitrary choice of model for π-meson on the one hand and of the doubtful validity of the perturbation method on the other hand. As for the first point, we shall discuss all the four possible models for π-meson to eliminate the arbitrariness. But, as for the second point, it is admitted to neglect damping and reactive corrections, because near the threshold energy for π-meson production mesons are emitted from loosely bounded parts of their clouds about the nucleons. In § 2 we shall give a general formula for collision cross-sections in a completely relativistic form, and in § 3 explanations for the approximation of our calculation will be given. From § 4 to § 7 cross-sections for all the four models are calculated and in the last section (§ 8) we shall compare them with each other.

§ 2. General formula for cross-sections

Since the calculations are much simplified in a completely relativistic form,* we combine Møller's formula for collision cross-sections and Feynman-Dyson's diagram method for calculating S-matrices. We consider the most general collision process as follows: two particles collide with each other and then n particles including the original ones are produced. From Møller's formula the cross-section is

$$\sigma=\frac{4\pi^2}{\mathfrak{B}}\prod_{i=1}^{n}\int\frac{d\boldsymbol{k}_i}{w_i}\delta(\boldsymbol{K}-\boldsymbol{K}^0)\delta(W-W^0)\,|(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n|\,I\,|\boldsymbol{k}_1^0,\boldsymbol{k}_2^0)|^2 \qquad (2.1)$$

with

$$\mathfrak{B}=\sqrt{|\boldsymbol{k}_1^0 w_2^0-\boldsymbol{k}_2^0 w_1^0|^2-|\boldsymbol{k}_1^0\times\boldsymbol{k}_2^0|^2}. \qquad (2.2)$$

$(\boldsymbol{k}_1^0, w_1^0)$ and $(\boldsymbol{k}_2^0, w_2^0)$ are energy momentum four vectors of incident particles,

* C. Morette has already computed in a relativistically invariant manner the cross-section for meson production by nucleon-nucleon collisions and we are much indebted to her manuscript.

respectively, and $(\boldsymbol{k}^i, w^i)$ is the same for the i-th particle produced. $(\boldsymbol{K}^0, W^0)$ and $(\boldsymbol{K}, W)$ are total energy momentum four vectors before and after the collision, respectively. $\mathfrak{B}$ results from the relativistic generalization of relative velocity of incident particles. The relation between a transition matrix $(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n | I | \boldsymbol{k}_1^0, \boldsymbol{k}_2^0)$ and a corresponding S-matrix in momentum space is

$$(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n | S | \boldsymbol{k}_1^0, \boldsymbol{k}_2^0) = \delta(\boldsymbol{K}-\boldsymbol{K}^0)\delta(W-W^0)\frac{(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n | I | \boldsymbol{k}_1^0, \boldsymbol{k}_2^0)}{\sqrt{w_1 \cdots w_n}\quad \sqrt{w_1^0 w_2^0}}. \tag{2.3}$$

From (2.3), we can readily find a way to calculate I-matrices relativistically with slight modifications of Feynman-Dyson's diagram method for S-matrices. As an illustration we assume particles to be either nucleons or mesons.

The wave function of a nucleon and the adjoint operator for it are as follows:

$$\psi_\alpha(x) = \sqrt{1/2(2\pi)^3}\sum_{r=1,2}\int d\boldsymbol{p}/\sqrt{m^2+\boldsymbol{p}^2}\,(\psi_r^+(\boldsymbol{p})u_{r,\alpha}^+(\boldsymbol{p})e^{-ip_\lambda x_\lambda} + \bar{\psi}_r^-(\boldsymbol{p})\bar{u}_{r,\alpha}^-(\boldsymbol{p})e^{+ip_\lambda x_\lambda}), \tag{2.4}$$

$$\bar{\psi}_\alpha(x) = \sqrt{1/2(2\pi)^3}\sum_{r=1,2}\int d\boldsymbol{p}/\sqrt{m^2+\boldsymbol{p}^2}\,(\bar{\psi}_r^+(\boldsymbol{p})\bar{u}_{r,\alpha}^+(\boldsymbol{p})e^{ip_\lambda x_\lambda} + \psi_r^-(\boldsymbol{p})u_{r,\alpha}^-(\boldsymbol{p})e^{-ip_\lambda x_\lambda}). \tag{2.5}$$

Here m is the nucleon mass. $r(=1, 2)$ and $+, -$ distinguish the direction of spin and the sign of energy, respectively. (We use the natural unit system $\hbar=c=1$ and the metric tensor $g_{11}=g_{22}=g_{33}=g_{44}=-1$ throughout.) Commutation relations for these decomposed fields are

$$\begin{cases}\{\psi_r^+(\boldsymbol{p}),\ \bar{\psi}_{r'}^+(\boldsymbol{p}')\}_+ = \{\psi_r^-(\boldsymbol{p}),\ \bar{\psi}_{r'}^-(\boldsymbol{p}')\}_+ = \delta_{rr'}\sqrt{m^2+\boldsymbol{p}^2}\,\delta(\boldsymbol{p}-\boldsymbol{p}'), \\ \text{remaining anti-commutators}=0. \end{cases} \tag{2.6}$$

Spinor wave functions u satisfy the following relations:

$$\sum_{r=1,2} u_{r,\alpha}^+(\boldsymbol{p})\bar{u}_{r,\beta}^+(\boldsymbol{p}) = (\gamma_\mu p_\mu + m)_{\alpha\beta} \text{ and } \sum_{r=1,2}\bar{u}_{r,\alpha}^-(\boldsymbol{p})u_{r,\beta}^-(\boldsymbol{p}) = (\gamma_\mu p_\mu - m)_{\alpha\beta} \tag{2.7}$$

with $\gamma_4=\beta$ and $\boldsymbol{\gamma}=\beta\boldsymbol{\alpha}$.

Similarly for symmetrical meson wave functions,

$$\phi^i(x) = \sqrt{1/2(2\pi)^3}\int d\boldsymbol{k}/\sqrt{\mu^2+\boldsymbol{k}^2}\,(\phi^{i*}(\boldsymbol{k})e^{ik_\lambda x_\lambda} + \phi^i(\boldsymbol{k})e^{-ik_\lambda x_\lambda})\ (i=1, 2, 3), \tag{2.8}$$

(for mesons with spin 0)

or

$$\phi_\mu^i(x) = \sqrt{1/2(2\pi)^3}\sum_{r=1}^{3}\int d\boldsymbol{k}/\sqrt{\mu^2+\boldsymbol{k}^2}\,v_\mu^r(\boldsymbol{k})\,(\phi_r^{i*}(\boldsymbol{k})e^{ik_\lambda x_\lambda} + \phi_r^i(\boldsymbol{k})e^{-ik_\lambda x_\lambda}),\ (i=1, 2, 3) \tag{2.9}$$

(for mesons with spin 1)

where $v_\mu^r(k)$ is a polarization vector of a meson with spin 1 and satisfies

$$\sum_{r=1}^{3} v_\mu^r(\boldsymbol{k})v_{\mu'}^r(\boldsymbol{k}) = -\delta_{\mu\mu'} + (k_\mu k_{\mu'}/\mu^2). \tag{2.10}$$

commutation relations are

$$\begin{cases}[\phi^i(\boldsymbol{k}), \phi^{*i'}(\boldsymbol{k}')]=\delta^{ii'}\sqrt{\mu^2+\boldsymbol{k}^2}\,\delta(\boldsymbol{k}-\boldsymbol{k}'), \\ \text{remaining commutators}=0\end{cases} \tag{2.11}$$

for mesons with spin 0, and similar expressions for mesons with spin 1.

Then for each figure of Feynman-Dyson's diagram method,* there corresponds one transition matrix ($|I|$) which is obtained by the following procedure.

(I) For each free nucleon line $\sqrt{1/2(2\pi)^3}\,u^*(\boldsymbol{p})$ or expressions like that is substituted.

(II) For each free meson line $\sqrt{1/2(2\pi)^3}$ or $\sqrt{1/2(2\pi)^3}\,\varepsilon^*_\mu(\boldsymbol{k})$ is substituted according as the spin of the free meson is 0 or 1.

(III) About other elements of figures as internal lines and world points there is no difference between S- and I-matrices.

Thus we have obtained a completely relativistic procedure for calculating cross-sections and this will be used in the following sections.

§ 3. Approximation of calculation

We shall calculate the transition probability from a state in which we have two nucleons with momentum $\boldsymbol{p}^0$ and $\boldsymbol{q}^0$ to a state in which we have two nucleons with momentum $\boldsymbol{p}$ and $\boldsymbol{q}$ and a meson with momentum $\boldsymbol{k}$.** Hereafter, we assume that quantities with a symbol $\sim$ refer to the laboratory system and those with no symbol refer to the center of gravity system. The threshold energy E_t for π-meson production and the maximum energy ϵ_m of emitted mesons when the incident energy E is given are as follows:

$$\begin{cases}\tilde{E}_t=2\mu+\mu^2/2m\sim 290 \text{ Mev.} \\ \tilde{\epsilon}_m=E/2+\mu^2/4m+\sqrt{\tilde{E}/(2m+\tilde{E})}\,\sqrt{(\tilde{E}^2-\mu^2/4m)^2-\mu^2} \text{ (for the laboratory system)}\end{cases}$$

and

$$\begin{cases}E_t=2\mu\sim 280 \text{ Mev.} \\ \epsilon_m=(m\tilde{E}+\mu^2/2)/\sqrt{2m(2m+\tilde{E})} \quad \text{(for the center of gravity system).}\end{cases}$$

As $\epsilon \ll m$ near the threshold energy, we may neglect k compared with $p^0 \pm p$ so that $p^2=p^{02}-m\epsilon$.*** In what follows we calculate cross-sections in powers of ϵ/m and leave only the first non-vanishing term. Further we shall carry on our calculation in the center of gravity system, because it helps us very much.

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* A detailed account of Feynman-Dyson's diagram method for calculating S-matrices is described in the following paper: F. J. Dyson, Phys. Rev. **75**, (1949) 1736.

** Here we assume the nucleons as free and neglect effects of the target nucleus upon them. On these points we shall have some discussions in § 8.

*** See Foldy and Marshak, reference 1.

§ 4. Symmetrical pseudoscalar meson

(A) The interaction Hamiltonian density for symmetrical pseudoscalar mesons is

$$H(x)=f\bar{\psi}(x)\gamma_5\tau_i\psi(x)\phi^i(x), \tag{4.1}$$

where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$, f is the coupling constant* and τ_i's are charge operators. The Feynman-Dyson's diagrams for meson production now considered are given by Fig. (A), (B), (C), (D) and those which can be obtained from them by interchanging p with q.

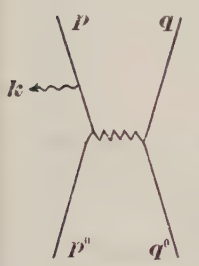

Fig. A

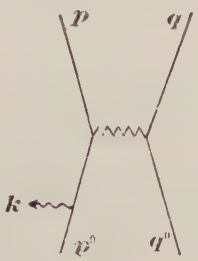

Fig. B

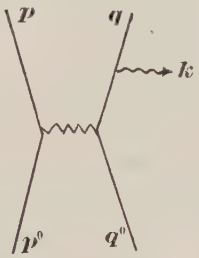

Fig. C

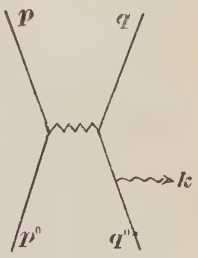

Fig. D

The transition matrices which correspond to these figures are summarized into the following expression:

$$\begin{aligned}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}\,|\,I\,|\,\boldsymbol{p}^0,\boldsymbol{q}^0)=&[1/2(2\pi)^3]^{5/2}(2\pi)^4f^3(\bar{u}^+(\boldsymbol{p})\gamma_5\tau_i u^+(\boldsymbol{p}^0))[1/(p_\nu-p_\nu{}^0)^2-\mu^2]\\&(\bar{u}^+(\boldsymbol{q})\{\gamma_5\tau_j(q_\lambda\gamma_\lambda+k_\lambda\gamma_\lambda-m)^{-1}\gamma_5\tau_i+\gamma_5\tau_i(q_\lambda{}^0\gamma_\lambda-k_\lambda\gamma_\lambda-m)^{-1}\gamma_5\tau_j\}u^+(\boldsymbol{q}^0))\\&-(\boldsymbol{pq})-(\boldsymbol{p}^0\boldsymbol{q}^0)+(\boldsymbol{pq},\boldsymbol{p}^0\boldsymbol{q}^0)\end{aligned} \tag{4.2}$$

where $(\boldsymbol{pq})$ or $(\boldsymbol{p}^0\boldsymbol{q}^0)$ are expressions which can be obtained from the previous one by interchanging p_λ with q_λ or $p_\lambda{}^0$ with $q_\lambda{}^0$, respectively, and $(\boldsymbol{pq},\boldsymbol{p}^0\boldsymbol{q}^0)$ by interchanging p_λ with q_λ and $p_\lambda{}^0$ with $q_\lambda{}^0$ simultaneously.

Taking the absolute square of $(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}\,|\,I\,|\,\boldsymbol{p}^0,\boldsymbol{q}^0)$, summing over the final spin states of the nucleons and averaging over the initial spin states, we find in the first approximation

$$|(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}|I|\boldsymbol{p}^0,\boldsymbol{q}^0)|^2=\frac{1}{2}\left(\frac{1}{2\pi}\right)^7f^6\left[\frac{\gamma}{(\boldsymbol{p}^0-\boldsymbol{p})^2}-\frac{\gamma'}{2}\frac{m\epsilon}{(\boldsymbol{p}^0-\boldsymbol{p})^2(\boldsymbol{p}^0+\boldsymbol{p})^2}\right], \tag{4.3}$$

where γ and γ' are numbers which are listed in Table 1. Mesons are emitted spherical-symmetrically and their energy distribution is given by the following differential cross-section

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* Cross-sections for symmetric pseudoscalar mesons with pseudovector coupling are by a factor of order ϵ/m smaller than those for symmetric pseudoscalar mesons with pseudoscalar coupling, so we only consider the latter case. By the same reason we shall neglect interaction Hamiltonian densities which contain derivatives of meson field's operator for other types of mesons.

$$d\sigma=(1/m^2)(f^2/4\pi)^3[\gamma f(\epsilon)-(\gamma'/2)g(\epsilon)]d\epsilon, \tag{4.4}$$

with

$$\begin{cases} f(\epsilon)=\dfrac{4\sqrt{\epsilon^2-\mu^2}}{m\epsilon_m}\tanh^{-1}\sqrt{\dfrac{\epsilon_m-\epsilon}{\epsilon_m}}, \\ g(\epsilon)=(\epsilon/2\epsilon_m-\epsilon)f(\epsilon). \end{cases} \tag{4.5}$$

In integrating the differential cross-section (4.4) analytically, we make the following approximations: we replace $\tanh^{-1}\sqrt{(\epsilon_m-\epsilon)/\epsilon_m}$ by $\sqrt{(\epsilon_m-\epsilon)/\epsilon_m}$ and $\sqrt{(\epsilon+\mu)/\epsilon_m}$ by a suitable average value $\langle\sqrt{(\epsilon+\mu)/\epsilon_m}\rangle_{AV.}$.

Then the total cross-section is

$$\sigma=(1/m^2)(f^2/4\pi)^3 \times[\gamma F-(\gamma'/2)G], \tag{4.6}$$

where

$$\begin{cases} F=\dfrac{\pi}{2}\left\langle\sqrt{\dfrac{\epsilon+\mu}{\epsilon_m}}\right\rangle_{AV.}\dfrac{(\epsilon_m-\mu)^2}{m\epsilon_m}, \\ G=8\pi\left\langle\sqrt{\dfrac{\epsilon+\mu}{\epsilon_m}}\right\rangle_{AV.}\left(\dfrac{3\epsilon_m-\mu}{2m}-\dfrac{\sqrt{\epsilon_m(2\epsilon_m-\mu)}}{m}\right)-F. \end{cases} \tag{4.7}$$

Table I. Values of α, β, γ and γ'.

Initial state	Final state	α	β	γ	γ'
p, p	p, p, π^0	0	0	1	2
	p, n, π^+	2	−2	2	2
p, n	p, p, π^-	1	1	1	1
	p, n, π^0	2	0	1	0
	n, n, π^+	1	1	1	1
n, n	p, n, π^-	2	−2	2	2
	n, n, π^0	0	0	1	2

(p and n indicate a proton and a neutron, respectively, and π^+, π^- and π^0 indicate a positive, negative and neutral meson, respectively.)

(B) Foldy nd Marshak have treated meson production as a second order process in which one step consists of the creation of a meson by one of the nucleons and the other step consists of the mutual scattering via the phenomenologically determined nuclear potential between them. Following their procedure, we take a nuclear potential V which is given by

$$V=\sum_{i=1}^{4}V_i \tag{4.8}$$

with

$$\begin{cases} V_1=(g_1^2/4\pi)\int dx dx'\bar{\psi}(x)\psi(x)\exp(-\mu r)/r\bar{\psi}(x')\psi(x'), \\ V_2=(g_2^2/4\pi)\int dx dx'\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x)\exp(-\mu r)/r\bar{\psi}(x')\gamma_5\gamma_\mu\psi(x'), \\ V_3=(g_3^2/4\pi)\int dx dx'\bar{\psi}(x)\tau_i\psi(x)\exp(-\mu r)/r\bar{\psi}(x')\tau_i\psi(x'), \\ V_4=(g_4^2/4\pi)\int dx dx'\bar{\psi}(x)\tau_i\gamma_5\gamma_\mu\psi(x)\exp(-\mu r)/r\bar{\psi}(x')\tau_i\gamma_5\gamma_\mu\psi(x'), \end{cases} \tag{4.11}$$

where $r=|x-x'|$ and $x_4=x_4'$.

Though we write spin matrices in a relativistically invariant form and assume the pseudoscalar coupling between mesons and nucleons instead of the pseudovector one, it will bring no difference to the cross-section, because in our approximation

we treat nucleons non-relativistically and the equivalence theorem between the pseudoscalar and the psudovector coupling holds.*

The angular distribution of mesons is spherically symmetric and differential cross-section becomes

$$d\sigma=\frac{u}{m^2}\left(\frac{f^2}{4\pi}\right)\left\{\left[\left(\frac{g_3^2-g_4^2}{4\pi}\right)^2+\left(\frac{g_2^2-g_4^2}{4\pi}\right)^2\right]f(\epsilon)-\left(\frac{g_3^2-g_4^2}{4\pi}\right)\left(\frac{g_2^2-g_4^2}{4\pi}\right)g(\epsilon)\right\}d\epsilon,** \tag{4.10}$$

with

$$\begin{cases} f(\epsilon)=2^6\sqrt{(\epsilon_m-\epsilon)/\epsilon_m}\ (\epsilon^2-\mu^2)^{3/2}(1/\epsilon)^4. \\ g(\epsilon)=2^7\dfrac{(\epsilon^2-\mu^2)^{3/2}}{\epsilon_m\epsilon^2(2\epsilon_m-\epsilon)}\tanh^{-1}\sqrt{\dfrac{\epsilon_m-\epsilon}{\epsilon_m}}. \end{cases} \tag{4.11}$$

The total cross-section is

$$\sigma=\frac{u}{m^2}\left(\frac{f^2}{4\pi}\right)\left\{\left[\left(\frac{g_3^2-g_4^2}{4\pi}\right)^2+\left(\frac{g_2^2-g_4^2}{4\pi}\right)^2\right]F-\left(\frac{g_3^2-g_4^2}{4\pi}\right)\left(\frac{g_2^2-g_4^2}{4\pi}\right)G\right\} \tag{4.12}$$

with

$$\begin{cases} F=2^5\pi\left\langle\sqrt{\dfrac{\epsilon+\mu}{\epsilon_m}}\right\rangle_{AV.}\dfrac{(\sqrt{\epsilon_m}-\sqrt{\mu})^2}{\sqrt{\epsilon_m\mu}}\left[1-\dfrac{(\epsilon_m+\mu)(\sqrt{\epsilon_m}+\sqrt{\mu})^2}{8\epsilon_m^2}\right], \\ G=2^7\pi\left\langle\sqrt{\dfrac{\epsilon+\mu}{\epsilon_m}}\right\rangle_{AV.}\left[\dfrac{3\epsilon_m-\mu}{2\epsilon_m}-\sqrt{\dfrac{2\epsilon_m-\mu}{\epsilon_m}}-\dfrac{1}{4}\left(\dfrac{\mu}{\epsilon_m}\right)^{3/2}\left(1-\dfrac{\sqrt{\mu(2\epsilon_m-\mu)}}{\epsilon_m}\right)\right]. \end{cases} \tag{4.13}$$

If we assume instead of the expression (4.8), that

$$V=(f^2/4\pi)\int dxdx'\bar{\psi}(x)\gamma_5\tau_i\psi(x)\exp(-\mu r)/r\bar{\psi}(x')\gamma_5\tau_i\psi(x'), \tag{4.14}$$

which can be deduced from the symmetrical pseudoscalar meson theory. Then the cross-section is just the same with (4.4) in (A).

§ 5. Symmetrical pseudovector meson

The interaction Hamiltonian density is

$$H(x)=if\bar{\psi}(x)\gamma_5\gamma_\mu\tau_i\psi(x)\phi_\mu^i(x). \tag{5.1}$$

The method of calculation which has just been used for symmetric pseudoscalar mesons can be used in the case of symmetric pseudovector mesons, and in the first approximation in which the nucleons are treated non-relativistically the angular distribution of mesons emitted is almost spherically symmetric but not exactly so. Here we write down only the differential cross-section.

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* The equivalence theorem between the pseudoscalar and the pseudovector coupling only holds for the case (B) and not for the case (A).

** This expression differs from the expression (23) of reference (1) in the sign of g-term.

$$d\sigma=\frac{m^2}{\mu^2}\left(\frac{f^2}{4\pi}\right)^3\left[\left(\frac{a}{2}+2\gamma+\frac{\gamma'}{8}\right)f(\epsilon)-\left(\frac{a}{2}-\frac{3}{32}\gamma'\right)g(\epsilon)\right]d\epsilon, \tag{5.2}$$

with

$$\begin{cases} f(\epsilon) = 2^6\sqrt{\dfrac{\epsilon_m-\epsilon}{\epsilon_m}}\,\dfrac{(\epsilon^2+2\mu^2)\sqrt{\epsilon^2-\mu^2}}{3\epsilon^2\mu^2}, \\ g(\epsilon)=\dfrac{\mu^2}{\epsilon_m(2\epsilon_m-\epsilon)}\ , \end{cases} \tag{5.3}$$

where a, γ and γ' are listed in Table 1.

§ 6. Symmetrical scalar meson

(A) The interaction Hamiltonian density is

$$H(x)=f\bar{\psi}(x)\tau_s\psi(x)\phi_s(x). \tag{6.1}$$

In the first order of ϵ/m the cross-section vanishes by the interference between four matrix-elements which come from Fig. (A), (B), (C) and (D), respectively. So we must proceed to the next approximation, in which relativistic effects is expected to be important.

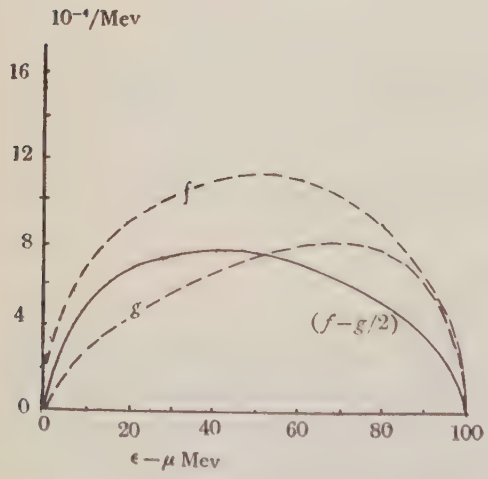

Fig. 1. Energy spectrum for pseudoscalar mesons (A) (Eq. (4.4)).
$f+(g/2)$: π^+ (or π^-) production by nucleon-nucleon collisions.

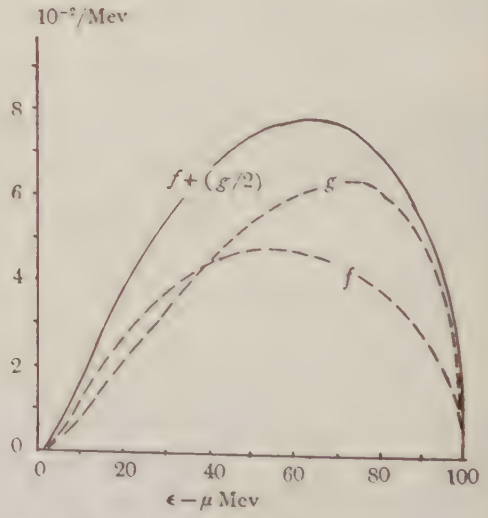

Fig. 2. Energy spectrum for pseudoscalar mesons (B) (Eq. (4.10)).
$f+(g/2)$: π^+ (or π^-) production by nucleon-nucleon collisions via Serber potential.

The differential cross-section is

$$d\sigma=(1/m^2)\,(f^2/4\pi)^3[\alpha f(\epsilon)-\beta g(\epsilon)]d\epsilon \tag{6.2}$$

with

$$f(\epsilon)=\sum_{i=1}^{3} f_i(\epsilon),\quad g(\epsilon)=\sum_{i=1}^{3} g_i(\epsilon). \tag{6.3}$$

That is, $f(\epsilon)$ and $g(\epsilon)$ are composed of three parts, each of which has a different origin.

(I) As a momentum transfer between the nucleons is not exactly the same in the process (A) (or (B)) as in the process (C) (or (D)), the remaining fraction which evades cancellation gives cross-section of the order of ϵ/m. $f_1(\epsilon)$ and $g_1(\epsilon)$ come from this effect and are given by

$$\begin{cases} f_1(\epsilon)=\dfrac{2^8}{3}\sqrt{\dfrac{\epsilon_m-\epsilon}{\epsilon_m}}\,\dfrac{m(2\epsilon_m-\epsilon)\,(\epsilon^2-\mu^2)^{3/2}}{\epsilon^6}, \\ g_1(\epsilon)=(f_1(\epsilon)/4)\,(\epsilon/2\epsilon_m-\epsilon)^3(1+K(\epsilon)), \end{cases} \tag{6.4}$$

where

$$K(\epsilon)=\frac{\epsilon^2}{(2\epsilon_m-\epsilon)\sqrt{\epsilon_m(\epsilon_m-\epsilon)}}\tanh^{-1}\sqrt{\frac{\epsilon_m-\epsilon}{\epsilon_m}}.$$

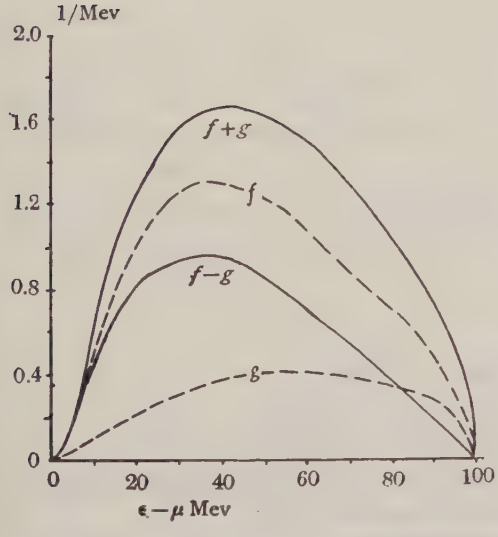

Fig. 3. Energy spectrum for scalar meson (A) (Eq. (6.2)).

$f+g$: π^+ (or π^-) production by $P-P$ (or $N-N$) collision.

$f-g$: π^+ (or π^-) production by $P-N$ collision.

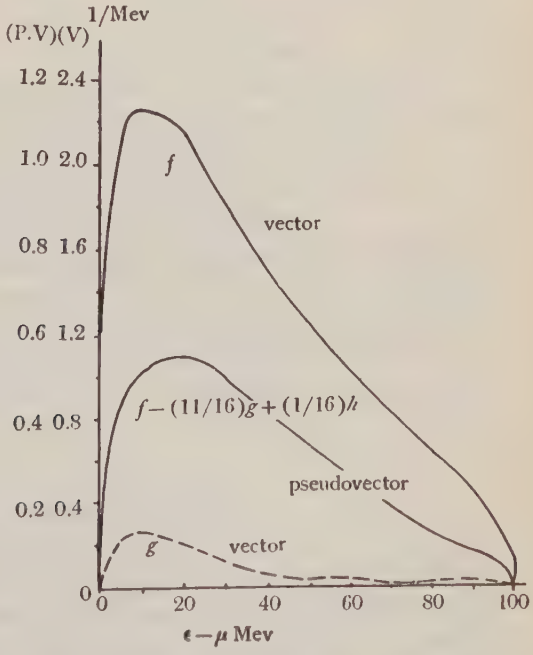

Fig. 4. Energy spectra for vector (Eq. (7.2)) and pseudovector mesons (Eq. (5.2)).

f: π^+ (or π^-) production for the vector theory. (Because g can be neglected compared with f.)

$(21/8)f-(13/32)g$: π^+ (or π^-) production for the *p.v.* theory.

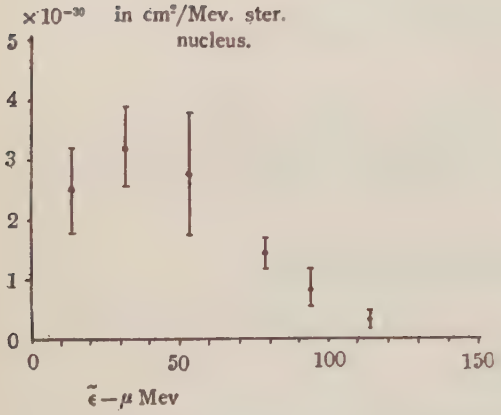

Fig. 5. The experimental energy spectrum at 90°.

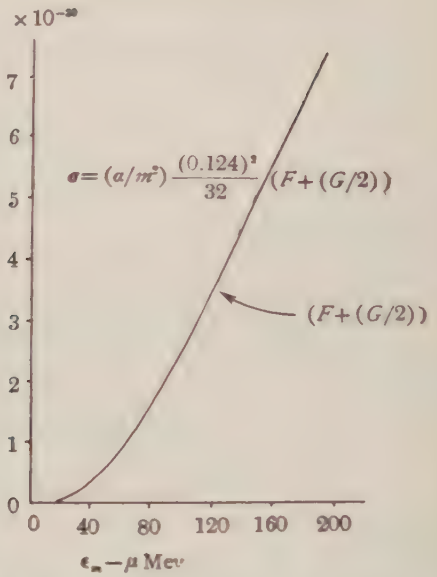

Fig. 6. The total cross-section for production of $\pi^{\pm}$ mesons (B).
(Here we assume Serber potential.)

(II) The second part of the cross-section is obtained by considering the nucleons recoil effect:

$$\begin{cases} f_2(\epsilon) = (f_1(\epsilon)/2)[1-(K(\epsilon)/8)], \\ g_2(\epsilon) = (f_1(\epsilon)/4)(\epsilon/2\epsilon_m-\epsilon)[1+(5K(\epsilon)/8)]. \end{cases} \tag{6.5}$$

(III) The interference effect between (I) and (II) makes the third contribution:

$$f_3(\epsilon) = f_1(\epsilon)[1+(1/16)(\epsilon/2\epsilon_m-\epsilon)^2(1+K(\epsilon)+2L(\epsilon))],$$

$$g_3(\epsilon) = (f_1(\epsilon)/2)(\epsilon/2\epsilon_m-\epsilon)[1+(1/4)(\epsilon/2\epsilon_m-\epsilon)(1+K(\epsilon)+2L(\epsilon))], \tag{6.6}$$

where

$$L(\epsilon) = \frac{\epsilon}{\sqrt{\epsilon_m(\epsilon_m-\epsilon)}} \tanh^{-1}\sqrt{\frac{\epsilon_m-\epsilon}{\epsilon_m}}.$$

The exchange effects,* however, are still of higher order in ϵ/m by the interference between four matrix-elements and may be neglected. The angular distribution of mesons differs from spherically symmetric one.

* Exchange terms are always smaller than ordinary terms for the symmetric scalar, vector, pseudoscalar or pseudovector theory. So if we assume a potential which is deduced from the meson theory and calculate the cross-section for meson production as a second order process, we shall have the same results.

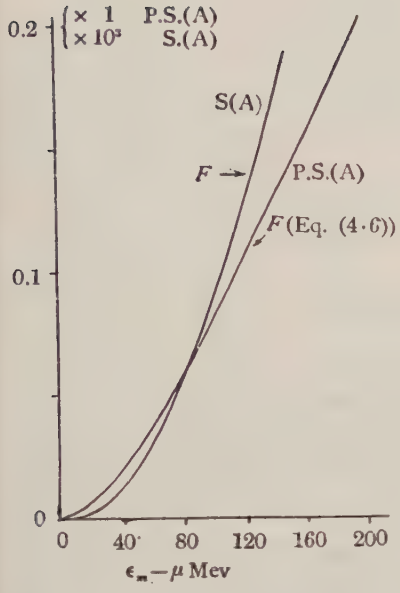

Fig. 7. The total cross-sections for production of S(A) and P.S. (A) mesons.

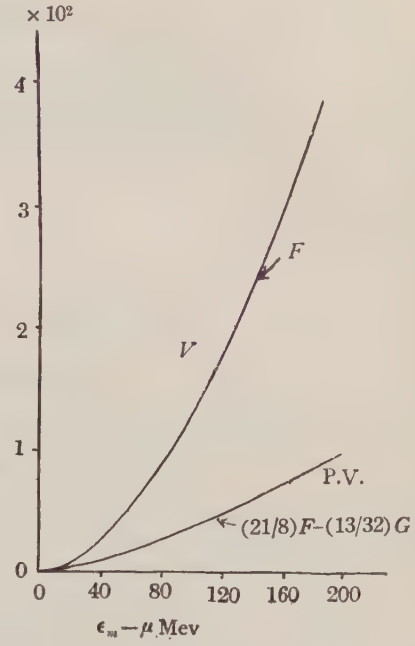

Fig. 8. The total cross-sections for production of Vector and P.V. mesons.

(In Fig. 6, 7 and 8 we plot only the average total cross-sections for the incident proton over the nucleons in the target nucleus having the equal numbers of protons and neutrons.)

The energy distribution curve of mesons and the total cross-section for π-meson production are plotted in Fig. 3 and Fig. 7, respectively.

(B) Next, as in the pseudoscalar case, we assume the phenomenological nuclear potential V ((4.8)) and treat meson production as a second order process. The differential cross-section is composed of three parts, each of them corresponding to effect the (I), (II) or (III) in the case of (A), and is given by

$$d\sigma = \sum_{i=1}^{3} d\sigma_i \tag{6.7}$$

with

$$d\sigma_1 = \frac{1}{m^2}\left(\frac{f^2}{4\pi}\right)[aC_1 f_1(\epsilon) - \beta C_2 g_1(\epsilon)]d\epsilon, \tag{6.8}$$

$$\begin{aligned} d\sigma_2 = \frac{1}{m^2}\left(\frac{f^2}{4\pi}\right)\Big[&\frac{a}{2} C_1 f_1(\epsilon)\left(1 - \frac{1}{2}K(\epsilon)\right) \\ &+ \frac{a}{16}(C_3 - 2C_4) f_1(\epsilon) K(\epsilon) - \frac{\beta}{4} C_5 f_1(\epsilon)\frac{\epsilon}{2\epsilon_m - \epsilon} \\ &- \frac{\beta}{8}\left(C_2 - \frac{C_6}{4}\right) f_1(\epsilon) K(\epsilon)\frac{\epsilon}{2\epsilon_m - \epsilon}\Big], \end{aligned} \tag{6.9}$$

$$d\sigma_3=\frac{1}{m^2}\left(\frac{f^2}{4\pi}\right)\left[aC_1f_1(\epsilon)+\frac{\beta}{2}C_5f_1(\epsilon)\frac{\epsilon}{2\epsilon_m-\epsilon}\right.$$
$$\left.-\left(\frac{a}{16}C_4+\frac{\beta}{8}C_2\right)f_1(\epsilon)\,(1+K(\epsilon)+2L(\epsilon))\right]d\epsilon, \tag{6.10}$$

where

$$\begin{aligned}
&(4\pi)^2C_1=g_3^4+3g_4^4,\\
&(4\pi)^2C_2=g_3^4+6g_3^2g_4^2-3g_4^4,\\
&(4\pi)^2C_3=(g_1^2-g_3^2)^2+3(g_2^2-g_4^2)^2,\\
&(4\pi)^2C_4=(g_1^2-g_3^2)g_3^2+3(g_2^2-g_4^2)g_3^2+3(g_1^2-g_3^2)g_4^2-3(g_2^2-g_4^2)g_4^2,\\
&(4\pi)^2C_5=g_3^2(g_3^2-g_1^2)+3g_4^2(g_4^2-g_2^2),\\
&(4\pi)^2C_6=(g_1^2-g_3^2)^2+6(g_1^2-g_3^2)(g_2^2-g_4^2)-3(g_2^2-g_4^2)^2.
\end{aligned} \tag{6.11}$$

With $g_3=f$ and $g_i=0$ $(i\neq 3)$, the expression (6.7) agrees with the expression (6.2) as is expected.

§ 7. Symmetrical vector meson

The interaction Hamiltonian density is

$$H(x)=f\bar{\psi}(x)\gamma_\mu\tau_i\psi(x)\phi_\mu^i(x), \tag{7.1}$$

and in the first approximation of treating nucleons non-relativistically the cross-section vanishes.

The differential cross-section is

$$d\sigma=(1/m^2)\,(f^2/4\pi)^2[af(\epsilon)-\beta g(\epsilon)] \tag{7.2}$$

with $f(\epsilon)=\sum_{i=1}^{3}f_i(\epsilon)$ and $g(\epsilon)=\sum_{i=1}^{3}g_i(\epsilon)$.

Functions $f_i(\epsilon)$ and $g_i(\epsilon)$ are related to those of symmetrical scalar mesons as follows:

$$\begin{cases}
f_1^v(\epsilon)=(\epsilon^2/\mu^2)f_1^s(\epsilon), \quad f_2^v(\epsilon)=(3\epsilon^2/\epsilon^2-\mu^2)f_2^s(\epsilon),\\
f_3^v(\epsilon)=-\sum_{i=1}^{3}f_i^s(\epsilon).
\end{cases} \tag{7.3}$$

and similar relations between $g^v(\epsilon)$ and $g^s(\epsilon)$, using suffices v and s which indicate vector and scalar mesons, respectively.

The angular distribution of mesons which are emitted by the effects of the nucleon's recoil (f_2^v and g_2^v) is spherically symmetric unlike that part of scalar mesons. Since this effect is the most predominent among other contributions in the low energy region on account of a factor $(3\epsilon^2/\epsilon^2-\mu^2)$, we shall have in practice a sherically symmetric angular distribution for low energy mesons as those obtained artificially.

§ 8 Discussion

Comparison with experiments of the predicted angular distribution, energy spectrum and total cross-section for production of π-mesons will throw light on certain properties of the π-mesons and the strength of its coupling with nucleons. But at present experimental deta are very few and inaccurate, so detailed analysis must be put off and here we satisfy ourselves only with summarizing and comparing our results each other.

(1) *Angular distribution*

The angular distribution curve of π-mesons produced can be expressed in our approximation as follows:

$$D(\theta)=1+\varkappa(\epsilon)P_2(\cos\theta), \tag{8.1}$$

where $\varkappa(\epsilon)=0$ for pseudoscalar mesons, $\varkappa(\epsilon)\sim 0$ for pseudovector and vector mesons and $\varkappa(\epsilon)$ is a quantity of order 1 for scalar mesons.

(2) *Energy spectrum*

If we assume collided particles are at rest initially, then $\epsilon_m\sim 163$ Mev for incident protons of energy$\sim$345 Mev. But as it is not the case, we make another assumption that nucleons in the target nucleus are moving in Fermi distribution. In consequence of these movements we shall have larger ϵ_m than$\sim$163 Mev, for example, $\sim$260 Mev for the nucleon which moves in the opposite direction to the incident proton with energy$\sim$25 Mev (maximum Fermi energy in carbon nucleus), i.e., with momentum$\sim$220 Mev. To compare directly our results with experimental data we must take the average over nucleons in the target nucleus. But here we only plot the predicted energy distribution curves** with $\epsilon_m\sim 240$ Mev together with the experimental curve***(Fig. 1, 2, 3, 4 and 5).

Near the lower end of the curve, i.e., when $\epsilon-\mu\ll\epsilon_m-\mu$, it increases in powers of $(\epsilon-\mu)^{1/2}$ for pseudoscalar (A), pseudovector and vector mesons and in powers of $(\epsilon-\mu)^{3/2}$ for pseudoscalar (B) and scalar mesons. Near the upper end it decreases in powers of $(\epsilon_m-\epsilon)^{1/2}$ for any case.

(3) *Total cross-section*

The total cross-sections near threshold energy are plotted in Fig. 6, 7 and 8. They increase in powers of $(\epsilon_m-\mu)^2$ for pseudoscalar (A), pseudovector and vector mesons and in powers of $(\epsilon_m-\mu)^3$ for pseudoscalar (B) and scalar mesons.

---

* If we consider other nucleons in the target nucleus, some transitions might be forbidden by Pauli principle. But as was shown by Feynman we have only to take account of this effect in intermediate states. For incident protons with energy$\sim$345 Mev recoil nucleons have energies larger than the maximum Fermi energy, so it will be admitted to neglect this effect.

** For pseudoscalar (B) and for scalar mesons (B) we take Serber potential, i.e., $g_1^2/4\pi=(3c_1+3c_3)/16$, $g_2^2/4\pi=(-3c_1+c_3)/16$ $g_3^2/4\pi=(c_1-3c_3)/16$ and $g_4^2/4\pi=(-c_1-c_3)/16$, where $c_1=0.280$ and $c_3=0.404$. See referenc

*** In this experiment meson nich were emitted in the direction perpendicular to that of incident protons were observed, and in this case we may put $\epsilon\sim\bar{\epsilon}$, approximately.

Table II. Numerical values of total cross-sections

Meson type	Pseudoscalar		Pseudovector	Scalar	Vector
$(f^2/4\pi)$	$0.05(2m/\mu)^2$		0.05	0.05	0.05
Incident energy		Total cross-section (cm^2)			
	(A)	(B)			
345 Mev	7.0×10^{-27}	4.7×10^{-31}	4.6×10^{-29}	2.2×10^{-31}	1.1×10^{-30}
500 Mev	3.5×10^{-26}	1.3×10^{-29}	4.7×10^{-27}	6.9×10^{-30}	1.1×10^{-29}

(These values are the average total cross-sections for production of charged mesons over the nucleons in the target nucleus having the equal numbers of protons and neutrons.)

As a preliminary estimation we give some results in Table II, assuming $f^2/4\pi\sim 0.05\ (2m/\mu)^2$ for pseudoscalar mesons and $f^2/4\pi\sim0.05$ for other mesons.

(III) *π^+ and π^- ratio*

If we assume that the incident nucleon is a proton and target nucleus contains equal numbers of protons and neutrons, then the π^-/π^+ ratio is given by

$R=1/3$ for pseudoscalar and pseudovector mesons,

$R<1/3$ for other cases which we have considered, because of the presence of β-term.

Of cource, it is true that we must take into consideration the effects of Coulomb potential of the target nucleus for low energy mesons, but it is to be noticed that R may differ from 1/3 by the presence of β-term.

In conclusion, we should like to express our sincerest thanks to Mr. Hayakawa, Mr. Yamaguchi and Mr. Fujimoto for their kind interest in this work.

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On the Generalized Transformation Function and the Integrability Condition.

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(Received April 10, 1950)

§ 1. Introduction

Recently Koba[1] has proposed a new method of calculating S-matrix, using only Lagrangian but not Hamiltonian, i.e., in the conventional field theory the law of motion is described by Tomonaga-Schwinger equation based on the Hamiltonian formalism, while S-matrix which is the integral of Tomonaga-Schwinger equation and calculated by Dyson's method[2] requires only Lagrangian. Therefore, so long as we employ the integral formalism as the basis of field theory, we need only Lagrangian and the role of Hamiltonian is virtual. The proof will be given in the following sections.

§ 2. The chronological operator P^*

In the following discussion we always use the interaction representation, the field quantities satisfy the same equations and commutation relations with these of free fields in the Heisenberg picture.

Now the total Lagrangian density L_{total} is decomposed into two parts:

$$L_{total}=L_{free}+L_{int}. \tag{2.1}$$

We discuss only about L_{int}, so we simply write it as L. Here L is a Lorentz invariant Hermite operator allowed to involve the derivatives of the field quantities of any order,[3] i.e., let u_α, u_α^* be the field quantities, then L is a polynomial or even a power series of the following quantities:

$$u_\alpha,\quad \frac{\partial u_\alpha}{\partial x_\mu},\quad \frac{\partial^2 u_\alpha}{\partial x_\mu \partial x_\nu},\quad \cdots;\quad u_\alpha^*,\quad \frac{\partial u_\alpha^*}{\partial x_\mu},\quad \frac{\partial^2 u_\alpha^*}{\partial x_\mu \partial x_\nu},\cdots .$$

Using such Lagrangian, we make an operator:

$$U[\sigma,\ \sigma_0]=\sum_{n=0}^{\infty} U_n[\sigma,\ \sigma_0], \tag{2.2}$$

where

$$U_n[\sigma,\ \sigma_0]=\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \int_{\sigma_0}^{\sigma} dx_1 \cdots \int_{\sigma}^{\sigma} dx_n\ P^*(L(x_1)\cdots L(x_n)). \tag{2.3}$$

Now we must define the notation P^* which is the modification of Dyson's P. But as it is hard to define it directly, we define all the matrix elements of $P^*(L(x_1)\cdots L(x_n))$.

In a quite similar way to the Dyson's diagram method, the matrix element of any process is given by the combination of the following terms:

$$<P^*\Big(\frac{\partial^l u_\alpha}{\partial x_{\mu 1}\cdots\partial x_{\mu l}},\ \frac{\partial^k u'^*_\beta}{\partial x'_{\nu 1}\cdots\partial x'_{\nu k}}\Big)>\ , \tag{2.4}$$

so that the general definition of P^* is reduced to the definition of (2.4), which is given by

$$<P^*\Big(\frac{\partial^l u_\alpha}{\partial x_{\mu 1}\cdots\partial x_{\mu l}},\ \frac{\partial^k u'^*_\beta}{\partial x'_{\nu 1}\cdots\partial x'_{\nu k}}\Big)>_0$$

$$=\frac{\partial^{l+k}}{\partial x_{\mu 1}\cdots\partial x_{\mu l}\ \partial x'_{\nu 1}\cdots\partial x'_{\nu k}}<P(u_\alpha,\ u'^*_\beta)>_0. \tag{2.5}$$

By this definition, the operator P^* commutes with the differential operator in contrast with the ordinary P, this is just the same with that proposed by Koba[3] and Matthews.[4]

P is determined by giving a family of space-like surfaces, and so long as $\sigma(x)\neq\sigma(x^1)$, $\varepsilon(x,\ x^1)$ commutes with the differential operator, so that in this case we know

$$<P^*(\quad)>_0=<P(\quad)>_0.$$

More generally speaking, if all the surfaces $\sigma(x_k)$ differ from each other, for instance

$$\sigma(x_1)>\sigma(x_2)>\ldots>\sigma(x_n),$$

(future > past)

then P and P^* coincide with each other:

$$P^*(L(x_1)\cdots L(x_n))=P(L(x_1)\cdots L(x_n)). \tag{2.6}$$

In the same way, if

$$\sigma_s>\sigma_t,\quad (s=1,\cdots,k;\ t=k+1,\cdots,n).$$

then P^* is decomposed into

$$P^*(L(x_1)\cdots L(x_n))=P^*(L(x_1)\cdots L(x_k))P^*(L(x_{k+1})\cdots L(x_n)). \tag{2.7}$$

What we must notice here is the fact that the quantities such as $<P(u_\alpha,\ u'^*_\beta)>_0$ are represented by Δ_F, S_F or their derivatives only which are independent of the choice of the family, so we can conclude:

$U_n[\sigma,\ \sigma_0]$, consequently $U[\sigma,\ \sigma_0]$ are determined uniquely by giving two space-like surfaces σ and σ.

§ 3. The generalized transformation function

Now suppose that the surface σ is deformed into σ' by an infinitesimal volume $d\omega$ at a point x on σ, then

$$
\begin{aligned}
&U_n[\sigma',\ \sigma_0]-U_n[\sigma,\ \sigma_0]\\
&=\binom{n}{1}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_n\ P^*(L(x_1)L(x_2)\cdots\cdots L(x_n))\\
&+\binom{n}{2}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2\int_{\sigma_0}^{\sigma}dx_3\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(L(x_1)L(x_2)\cdots L(x_n))\\
&+\cdots\cdots.
\end{aligned}
$$

If we employ $[\sigma_1,\ \sigma_2)$ as the integration domain of $\int_{\sigma_1}^{\sigma_2}$, then $[\sigma_0,\ \sigma,)$ $[\sigma,\ \sigma')$ have no common point, and so P^* is separated by (2.7), thus we get

$$
\begin{aligned}
&U_n[\sigma',\ \sigma_0]-U_n[\sigma,\ \sigma_0]\\
&=\left[\frac{i}{\hbar}\int_\sigma^{\sigma'}L(x_1)dx_1\right]\left[\frac{1}{(n-1)!}\left(\frac{i}{\hbar}\right)^{n-1}\int_{\sigma_0}^{\sigma}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(L(x_2)\cdots L(x_n))\right]\\
&+\left[\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2\ P^*(L(x_1)L(x_2))\right]\left[\frac{1}{(n-2)!}\left(\frac{i}{\hbar}\right)^{n-2}\right.\\
&\qquad\left.\int_{\sigma_0}^{\sigma}dx_3\cdots\int_{\sigma_0}^{\sigma}dx_n\ P^*(L(x_3)\cdots L(x_n))\right]\\
&+\cdots\cdots\\
&=\left[\frac{i}{\hbar}\int_\sigma^{\sigma'}L(x_1)dx_1\right]U_{n-1}[\sigma,\ \sigma_0]+\left[\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2\,P^*(L(x_1)L(x_2))\right]U_{n-2}[\sigma,\sigma_0]\\
&+\cdots\cdots.
\end{aligned}
\tag{3.1}
$$

We put:

$$
\begin{aligned}
&\lim_{\sigma'\to\sigma}\frac{1}{d\omega}\left[\frac{i}{\hbar}\int_\sigma^{\sigma'}L(x_1)dx_1\right]\equiv-\frac{i}{\hbar}H_1(x),\\
&\lim_{\sigma'\to\sigma}\frac{1}{d\omega}\left[\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2\ P^*(L(x_1)L(x_2))\right]\equiv-\frac{i}{\hbar}H_2(x),\\
&\qquad\cdots\cdots\cdots\cdots\\
&\lim_{\sigma'\to\sigma}\frac{1}{d\omega}\left[\frac{1}{r!}\left(\frac{i}{\hbar}\right)^r\int_\sigma^{\sigma'}dx_1\cdots\int_\sigma^{\sigma'}dx_r\ P^*(L(x_1)\cdots L(x_r))\right]\equiv-\frac{i}{\hbar}H_r(x).
\end{aligned}
\tag{3.2}
$$

In general, $H_2,\cdots$ do not vanish,[t] because of the existence of 4-dimensional δ-functions in the integrands, and as the integrands of (3.2) are family independent, H's are independent of the method of limiting process $\sigma'\to\sigma$ and completely determined by giving x and σ only.

Inserting (3.2) into (3.1), one obtains

$$\frac{\delta}{\delta\sigma(x)}U_n[\sigma,\ \sigma_0]=-\frac{i}{\hbar}\left\{H_1(x)U_{n-1}[\sigma,\ \sigma_0]+H_2(x)U_{n-2}[\sigma,\ \sigma_0]+\cdots\cdots\right\}. \tag{3.3}$$

Now put

$$H[x\,;\ \sigma]=\sum_{n=1}^{\infty}H_n(x). \tag{3.4}$$

Then summing (3.3) up with respect to n, we get

$$\frac{\delta}{\delta\sigma(x)}U\ [\sigma,\ \sigma_0]=-\frac{i}{\hbar}H[x\,;\ \sigma]\ U[\sigma,\ \sigma_0]. \tag{3.5}$$

Next in the same way with (3.3), one can derive

$$\frac{\delta}{\delta\sigma_0(x_0)}U_n[\sigma,\ \sigma_0]=\frac{i}{\hbar}\left\{U_{n-1}[\sigma,\ \sigma_0]H_1(x_0)+U_{n-1}[\sigma,\ \sigma_0]H_2(x_0)+\cdots\right\}. \tag{3.6}$$

Comparing (3.2) with (2.3), we see

$$H_r(x)=i\hbar\lim_{\substack{\sigma'\to\sigma\\ \sigma''\to\sigma}}\frac{1}{d\omega}U_r\ [\sigma',\ \sigma], \qquad (r=1,2,\cdots) \tag{3.7}$$

especially

$$H_1(x)=-L(x), \tag{3.8}$$

but the definition (3.7) is not symmetric with respect to future and past, so that we can equivalently define it as follows:

$$H_r(x)=\frac{i\hbar}{2}\lim_{\substack{\sigma'\to\sigma\\ \sigma''\to\sigma}}\frac{1}{d\omega}\left[U_r[\sigma',\ \sigma]+U_r[\sigma,\ \sigma'']\right]. \tag{3.9}$$

Then its functional derivative is given by

$$\frac{\delta}{\delta\sigma(x')}H_r\ (x)=\frac{i\hbar}{2}\lim_{\substack{\sigma'\to\sigma\\ \sigma''\to\sigma}}\frac{1}{d\omega}\left[\frac{\delta U_r[\sigma',\ \sigma]}{\delta\sigma(x')}+\frac{\delta U_r[\sigma,\ \sigma'']}{\delta\sigma(x')}\right],$$

inserting (3.3) and (3.6) into above equation, one easily obtains

$$\frac{\delta H_r(x)}{\delta\sigma(x')}=\frac{i}{2\hbar}\left\{[H_1(x),H_{r-1}(x')]+[H_2(x),H_{r-2}(x')]+\cdots\right\}. \tag{3.10}$$

This is the fundamental equation of our theory.

If $H_1,\ H_2,\ldots,H_{r-1}$ are Hermitian, then from (3.10) H_r is also Hermitian, while $H_1=-L$ is Hermitian, so that all H's are Hermitian by the mathematical induction. Next summing (3.10) up with respect to r, we get

$$\frac{\delta}{\delta\sigma(x')}H[x\,;\ \sigma]=\frac{1}{2\hbar}\left[H[x\,;\ \sigma],\ H[x'\,;\ \sigma]\right]. \tag{3.11}$$

This directly guarantees that the Hamiltonian (3.4) actually satisfies the integrability condition.

As $H[x\,;\,\sigma]$ is Hermitian, $U\,[\sigma,\,\sigma_0]$ is unitary from (3.5).

Thus translating all the above results into the property of $U\,[\sigma',\,\sigma]$, we get:

The property of $U[\sigma'\,,\sigma]$.

(1) $U\,[\sigma',\,\sigma]$ is an unitary operator uniquely determined by giving two space-like surfaces σ and σ'.

(2) $U\,[\sigma,\,\sigma]=1$.

(3) $U\,[\sigma'',\,\sigma']\;U\,[\sigma',\,\sigma].=U[\sigma'',\,\sigma]$.

Clearly these results show that $U\,[\sigma',\,\sigma]$ is the generalized transformation function, and S-matrix is readily given by

$$S=U[\infty,-\infty]. \tag{3.12}$$

§ 4. Density matrix

In this section, we will discuss the property of the density matrix in the interaction representation.

Consider the generalized Schroedinger equation:

$$i\hbar\frac{\delta\Psi\,[\sigma]}{\delta\sigma\,(x)}=H(x)\;\Psi[\sigma], \tag{4.1}$$

and expand the state vector $\Psi[\sigma]$,

$$\Psi[\sigma]=\sum_m a_m[\sigma]\;\Phi_m, \tag{4.2}$$

where Φ_m refer to some eigenvectors which are independent of σ, and satisfy

$$(\Phi_m,\;\Phi_n)=\delta_{mn},$$

and the coefficients $a_m[\sigma]$ are c-numbers.

Inserting (4.2) into (4.1), we get immediately

$$\begin{cases} i\hbar\dfrac{\delta a_m[\sigma]}{\delta\sigma\,(x)}=\sum\limits_n H_{mn}\,(x)a_n[\sigma], \\ i\hbar\dfrac{\delta a_m^*[\sigma]}{\delta\sigma(x)}=-\sum\limits_n a_n^*[\sigma]H_{nm}\,(x), \end{cases} \tag{4.3}$$

where

$$H_{mn}=(\Phi_m,\;H\,\Phi_n).$$

Now we define the density matrix as follows:

$$\rho(\sigma,\;\sigma')=(a_m[\sigma].\;a_n^*[\sigma']). \tag{4.4}$$

Then from (4.3), $\rho(\sigma,\,\sigma')$ satisfies the following equations:

$$\begin{cases} i\hbar \dfrac{\delta}{\delta\sigma(x)}\ \rho(\sigma,\ \sigma')=H(x)\rho(\sigma,\ \sigma'), \\ i\hbar \dfrac{\delta}{\delta\sigma'(x')}\ \rho(\sigma,\ \sigma')=-\rho(\sigma,\ \sigma')H(x'). \end{cases} \tag{4.5}$$

The expectation value of a physical quantity Q is given by

$$\begin{aligned} \langle Q\rangle=(\Psi[\sigma],\ Q\Psi[\sigma])&=\sum_{m,n} a_m^*[\sigma]\ a_n[\sigma]\ Q_{mn} \\ &=S_p(\rho(\sigma,\ \sigma)Q). \end{aligned}$$

We write simply

$$\rho(\sigma,\ \sigma)=\rho(\sigma),$$

then

$$\langle Q\rangle=S_p(\rho(\sigma)Q). \tag{4.6}$$

Next from the normalization condition:

$$(\Psi[\sigma],\ \Psi[\sigma])=\sum_m a_m^*[\sigma]\ a_m[\sigma]=1,$$

one easily obtains

$$\begin{aligned} (\rho(\sigma,\ \sigma')\rho(\sigma',\ \sigma''))_{kl}&=\sum_s a_k[\sigma]\ a_s^*[\sigma']\ a_s[\sigma']\ a_l^*[\sigma''] \\ &=a_k[\sigma]\ a_l^*[\sigma''], \end{aligned}$$

i.e.

$$\rho(\sigma,\ \sigma')\ \rho(\sigma',\ \sigma'')=\rho(\sigma,\ \sigma''). \tag{4.7}$$

especially

$$\rho(\sigma)^2=\rho(\sigma). \tag{4.8}$$

Solving (4.5), $\rho(\sigma,\ \sigma')$ is given by

$$\rho(\sigma,\ \sigma')=U[\sigma,\ \sigma_0]\ \rho(\sigma_0)\ U[\sigma_0,\ \sigma'], \tag{4.9}$$

which describes the law of motion for the density matrix by means of the generalized transformation function.

If we write for simplicity as

$$\rho(\sigma_0)=\rho_0,\ U[\sigma,\ \sigma_0]=S,$$

then (4.6) is rewritten as follows:

$$\langle Q\rangle=S_p(S\rho_0 S^{-1}\ Q). \tag{4.10}$$

The relation (4.8) means that the Hermite operator $\rho(\sigma)$ has the eigenvalues 0 and 1, thus we know the connection between the generalized transformation function and the density matrix, but here the concept of the density matrix is differently used from the ordinary one.

§ 5. Integral formalism

As has been proved in the previous section, only Lagrangian is required to calculate the generalized transformation function, and equivalent Hamiltonian always exists in principle, but we need only the existence of it, and not the concrete form.

Therefore we can start from the integral formalism, so we will translate the differential laws into the integral laws.

(1) The law of motion.

$$\left[H(x)+\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x)}\right]\Psi[\sigma]=0, \tag{1d.}$$

$$\Psi[\sigma]=U[\sigma,\ \sigma_0]\Psi[\sigma_0]. \tag{1i.}$$

(2) The conservation law.

Let $Q[\sigma]$ be a conservative quantity, then the following condition must be satisfied:

$$\left[H(x)+\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x)},Q(\sigma)\right]=0, \tag{2d.}$$

$$Q[\sigma]U[\sigma,\ \sigma_0]=U[\sigma,\ \sigma_0]\ Q[\sigma_0]. \tag{2i.}$$

(3) The supplementary condition.

In order that the supplementary condition:

$$\Xi[\sigma]\ \Psi[\sigma]=0$$

does not contradict to the law of motion, the following condition must be satisfied;

$$\left[H(x)+\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x)},\ \Xi[\sigma]\right]=0, \tag{3d.}$$

$$\Xi[\sigma]U[\sigma,\ \sigma_0]=U[\sigma,\ \sigma_0]\Xi[\sigma_0]. \tag{3i.}$$

An example of the interaction which is easier to introduce in the integral formalism than in the differential formalism is of a type as follows:

$$L(x)=\{f(\Box)j_\mu(x)\}\ A_\mu(x)$$
$$=\int j_\mu(\xi)\eta(\xi-x)\ A_\mu(x)\ d^4\xi,$$

where

$$\eta(x)=\frac{1}{(2\pi)^4}\int f(-k_\mu^2)e^{ik_\mu x_\mu}\ d^4k.$$

Thus even the non-local interaction can be treated in a covariant way in this integral formalism.

Appendix

In the text, we see that the Lagrangian method is more convenient than the Hamiltonian method for the calculation of S matrix, and that practically the former is the only one when canonical description is impossible. So it seems necessary to clarify the role of Lagrangian more deeply.

In the canonical formalism, Lagrangian is used to derive the field equations from the variation principle and also to quantize the field variables. While in our theory, the unitary use of total Lagrangian is impossible. For instance, the free part of the Lagrangian is utilized to quantize the field variables and to derive the equations that they satisfy in the same way with the canonical case.

Such a procedure is always possible since the field quantities in the interaction representation coincide with those of free fields in the Heisenberg representation whiche an be described in the canonical form.

Concretely speaking, we employ the representation in which field operators satisfy the following variation principle,

$$\delta\int L_{total}\, d^4x=0. \tag{A.1}$$

Thus the role of the free part in the interaction representation is completely equivalent to the total Lagrangian of the canonical formalism in the Heisenberg representation.

On the other hand, the interaction part of Lagrangian is considered to be the generating function of S matrix or the generalized transformation function that rule the causality.

The relation between the free and interaction parts of Lagrangian resembles to that between the tram and the rail.

On this stand point of view, S matrix is a functional of the interaction part of Lagrangian

$$S[L_{int.}]=\sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n\int_{-\infty}^{\infty}dx_1\ldots\int_{-\infty}^{\infty}dx_n\, P^*(L(x_1)\ldots L(x_n)). \tag{A.2}$$

The employment of the interaction representation is essential in our generalized theory.

In the canonical formalism, field equations in the Heisenberg representation can be derived in two ways.

The one is the variation principle:

$$\delta\int L_{total}\, d^4x=0, \tag{A.3}$$

which is independent of the canonical formalism.

While the other is based on the field quantization, after which we can construct the total Hamiltonian by the common rule of the canonical formalism. Then the equation of motion for an arbitrary field quantity Q is given by

$$\frac{dQ}{dt}=\frac{\partial Q}{\partial t}+\frac{i}{\hbar}[H_{total}\,,\ Q]. \tag{A.4}$$

As the characteristic feature of the canonical formalism, field equations derived in the both methods coincide.

But it is not true in our general case, as has been noticed by Koba.[1] Our method is equivalent to the second one essentially, the quantization, however, is referred only to the free part, but not the total Lagrangian.

Also the interaction part of Hamiltonian cannot be derived automatically from Lagrangian in contrast to the canonical case, for it should be constructed by the requirement of the integrability condition.

Corresponding to (A.4), we have

$$\frac{d}{dx_\mu}<Q(x)>=<\frac{\partial Q(x)}{\partial x_\mu}>+<\frac{i}{\hbar}\int^{\sigma}dS_\mu'[H_{int}[x';\ \sigma],\ Q(x)>, \tag{A.5}$$

where $<\ >$ means the expectation value on a surface σ, and field variable $Q(x)$ is expressed in the interaction representation. $H_{int}[x\,;\ \sigma]$ is non-local, but the spread is confined within the surface σ, so that the initial condition on a surface σ_0 is enough to determine the future behaviour of the fields system in agreement with the causality principle.

Whlie the field equations derived from the variation principle (A.3) in the non-canonical case have the non-local interaction which spreads not only space-likely on σ, but also time-likely,[1] so that the knowledge on the initial surface σ_0 is no more enough, and the future behaviour depends also on the past history in contrast to the basic causality principle of quantum mechanics.

Thus the unitary description by the total Lagrangian is broken and the variation principle does no more hold in our non-canonical theory.

This is a very gloomy conclusion, for theorems which can be derived from the variation principle have no validity and they should be demonstrated by another direct method.

For instance, gauge invariance is no more clear from the form of the total Lagrangian, since it requires an intimate connection between the free and interaction parts of Lagrangian.

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Integral Equations between Distribution Functions of Molecules

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§ 1. Introduction

We may define distribution functions for given number, n, of molecules as being proportional to the probability density of finding n molecules in a certain configurational position, averaged over all positions of the other molecules of the system. These functions are the exponent of minus the potential of average force between molecules, devided by kT. Mayer[1] derived a set of integral equations which relate variations in the potentials of average force at two different fugacities. On the other hand, Kirkwood[2] evolved an integral equation for the radial distribution function, and Born and Green[3] obtained a similar equation. But the relations between these integral equations and Mayer's general variational method are apparently not known. The purpose of this paper is to attempt to clarify these relations, at the same time, to derive new integral equation which is a generalization of the Born-Green equation. We will deal with a multicomponent system, because the final equations are in no way more complicated if a system of many components is considered than if a single component system is handled.

§ 2. Symbolism

In treating a problem involving the coordinates of an extremely large number of molecules it will be necessary to use a system of symbols originally introduced by Mayer[1] in order to attain some brevity of expression. Let us consider an infinite system containing σ types of molecular species, and use the subscript s (or t) to refer to a special species of type s(or t). The symbols without subscripts refer to a set of quantities

$$\rho=\rho_1,\ \rho_2,\cdots\cdots,\rho_s,\cdots\cdots,\rho_\sigma, \tag{1}$$

$$z=z_1,\ z_2,\cdots\cdots,z_s,\cdots\cdots,z_\sigma. \tag{2}$$

The symbol ρ_s indicates the density of molecules of species s, in molecules per unit volume, and z_s is the fugacity normalized in density unit.

The letters k, m, n, etc., when written without subscripts, will be used to indicate sets of integers:

$$k=k_1,\ k_2,\cdots\cdots,k_s,\cdots\cdots,k_\sigma, \tag{3}$$

of k_s molecules of type s. The short hand notation:

$$k!=k_1!\ k_2!\cdots k_s!\cdots k_\sigma!, \tag{4}$$

$$\rho^k=\rho_1^{k_1}\rho_2^{k_2}\cdots\rho_s^{k_s}\cdots\rho_\sigma^{k_\sigma} \tag{5}$$

$$z^k=z_1^{k_1}z_2^{k_2}\cdots z_s^{k_s}\cdots z_\sigma^{k_\sigma} \tag{6}$$

will be adopted.

The symbol

$$(i_s)=x_{i_s},\ y_{i_s},\ z_{i_s},\ q_{1i_s},\ q_{2i_s},\cdots \tag{7}$$

indicates the configurational coordinates of the i-th molecule of type s, and in general will include internal coordinates, q_{1i_s}, q_{2i_s},$\cdots$,as well as the Cartesian coordinates, x_{i_s}, y_{i_s}, z_{i_s}, of the center of mass. The coordinates of k_s molecules of type s will be represented by

$$\{k_s\}=(1),\ (2),\cdots,(k_s). \tag{8}$$

And we will denote the coordinates of all molecules of such a set k by

$$\{k\}=\{k_1\},\ \{k_2\},\cdots,\{k_s\},\cdots,\{k_\sigma\}. \tag{9}$$

The volume elements are similarly expressed by $d(i_s)$, $d\{k_s\}$, and $d\{k\}$.

A function, $\Psi_n\{n\}$, of the coordinates of the set n of molecules can be expressed as a sum of the component functions $\psi_\nu\{\nu\}_n$ which are functions of the coordinates $\{\nu\}_n$ of subsets of molecules out of the set n. These component functions are defined by either of two reciprocal sets of equations

$$\Psi_n=(\sum\{\nu\}_n)\psi_\nu, \tag{10}$$

or

$$\psi_\nu=(\sum\{n\}_\nu)(-)^{\nu-n}\Psi_n, \tag{11}$$

where the summation symbol, $(\sum\{\nu\}_n)$, indicates summation of functions of such subsets, $\{\nu\}_n$, over all subsets, for all subsets, for all values of ν, $0\leq\nu_s\leq n_s$, including a single term for which all $\nu_s=0$.

§ 3. Variational method

With this notation we now define functions $F_n(\rho,\ T,\ \{n\})$, of the coordinates of a set n of molecules, by the statement that in a system of infinite extent ($V\gg n_s/\rho_s$ for all s) at the temperature T, and density ρ, the probability that there are an appropriate set n of molecules at the coordinates $\{n\}$ within the configura-

tion element $d\{n\}$ is equal to $\rho^n F_n(\rho,T,\{n\})d\{n\}$. It is to be noted that with this definition the functions F_n are so normalized that,

$$\lim_{V\to\infty}\left[\frac{1}{V}\iint\cdots\int F_n\, d\{n\}\right]=1. \tag{12}$$

Mayer[4] and McMillan[5] have shown that for two sets of densities ρ_α and ρ_β, for which the pressures are p_α and p_β, and for which the fugacities are z_α and z_β, respectively, the distribution functions are related by two sets of equations:

$$G_n=\sum_{m\geq 0}\frac{z^m}{m!}\iint\cdots\int G^*_{n+m}d\{m\}, \tag{13}$$

or

$$G^*_n=\sum_{m\geq 0}\frac{(-z)^m}{m!}\iint\cdots\int G_{n+m}\, d\{m\}, \tag{14}$$

where

$$G_n=\left[\exp\frac{p_\alpha V}{kT}\right]\left[\frac{\rho_\alpha}{z_\alpha}\right]^n F_n(\rho_\alpha, T,\{n\}), \tag{15}$$

$$G^*_n=\left[\exp\frac{p_\beta V}{kT}\right]\left[\frac{\rho_\beta}{z_\beta}\right]^n F_n(\rho_\beta, T, \{n\}), \tag{16}$$

and

$$z=z_\alpha-z_\beta.$$

We will consider two sets of functions $G_n(y)$ and $G^*_n(y)$ which depend upon a parameter y. And we now assume that $G^*_n(y_0)=G^*_n$ for a certain value of the parameter, y_0, and that functions $G_n(y)$ depend on y only through the equation

$$G_n(y)=\sum_{m\geq 0}\frac{z^m}{m!}\iint\cdots\int G^*_{n+m}(y)\, d\{m\}. \tag{17}$$

Then, from (15) and (16), we have $G_n(y_0)=G_n$, and the reciprocal of (17) is

$$G^*_n(y)=\sum_{m\geq 0}\frac{(-z)^m}{m!}\iint\cdots\int G_{n+m}(y)\, d\{m\}. \tag{18}$$

It is necessary to assume that $G_m(y)$ and $G^*_m(y)$ are symmetrical in permutations of the coordinates of identical molecules except the coordinates of a certain molecule i_0. Then we cannot define unsymmetrical $G_{n0}(y)$ and $G^*_{n0}(y)$ unless the set n_0 contains the molecule i_0.

We now define the functions Ψ_n and Ψ^*_n by

$$\Psi_n=\frac{d\ \ln\ G_n(y)}{dy} \tag{19}$$

$$\Psi_n^* = \frac{d \ln G^*(y)}{dy} \tag{19'}$$

and the component functions ψ_ν and ψ_ν^* by (11) and

$$\psi_\nu^* = (\Sigma\{n\}_\nu)(-)^{\nu-n}\Psi_n^*, \tag{20}$$

so that (10), and,

$$\Psi_n^* = (\Sigma\{\nu\}_n)\psi_\nu^* \tag{21}$$

are valid. From these equations Mayer obtained

$$\psi_k = \sum_{m\geqq 0}\frac{z_m}{m!}(\sum\{\varkappa\}_k)\iint_V\cdots\int\psi_{k+m}^* K_{k,\varkappa,m}\, d\{m\}, \tag{22}$$

in which the kernels, $K_{k,\varkappa,m}$, are defined by the equation:

$$K_{k+n,n,m} = (\sum\{\varkappa\}_k)(-)^{k-\varkappa}\frac{G_{\varkappa+n+m}(y)}{G_{\varkappa+n}(y)} \tag{23}$$

By the same operation he obtained

$$\psi_k^* = \sum_{m\geqq 0}\frac{(-z)^m}{m!}(\sum\{\varkappa\}_k)\iint_V\cdots\int\psi_{k+m}K_{k,\varkappa,m}^*\, d\{m\}, \tag{24}$$

with

$$K_{k+n,n,m}^* = (\sum\{\varkappa\}_k)(-)^{k-\varkappa}\frac{G_{\varkappa+n+m}^*(y)}{G_{\varkappa+n}^*(y)}. \tag{25}$$

§ 4. Differentiation with respect to the coordinate of a molecule

Let us consider an arbitrary function, $\Phi_n\{n\}$, of the coordinates of the set n of molecules. And for the sake of covenience we will represent $\Phi_n\{n\}$ by $\Phi_s(i_s)$, $\Phi_{st}((i_s)(j_t))$ and $\Phi_{str}((i_s)(j_t)(k_r))$ when $n=(i_s)$, $n=(i_s)(j_t)$ and $n=(i_s)(j_t)(k_r)$ respectively. $K_{k+n,n,m}$ will be denoted by $K(\{k+n\};\{n\};\{m\})$. The potentials of average force between molecules n are defined by

$$W_n\{n\} = -kT\ln F_n(\rho_\alpha, T, \{n\}), \tag{26}$$

$$W_n^*\{n\} = -kT\ln F_n(\rho_\beta, T, \{n\}). \tag{26'}$$

The component potentials, $w_n\{n\}$, are defined by the equations

$$w_\nu\{\nu\} = (\sum\{n\}_\nu)(-)^{\nu-n}W_n\{n\}_\nu, \tag{27}$$

$$w_\nu^*\{\nu\} = (\sum\{n\}_\nu)(-)^{\nu-n}W_n^*\{n\}_\nu. \tag{27'}$$

If differentiation is performed with respect to the coordinates (i_s) of the i-th molecule of type s, we have, according to (15), (19) and (26),

$$\Psi_s(i_s) = -\frac{1}{kT}\frac{\partial W_s(i_s)}{\partial x_{is}}, \tag{28}$$

$$\Psi_{st}((i_s)(j_t)) = -\frac{1}{kT}\frac{\partial W_{st}((i_s)(j_t))}{\partial x_{i_s}}, \tag{29}$$

since ρ and p do not depend on (i_s).

From (28) and (29), it will be easily seen the functions $\Psi_n kT$ are the average forces. Ψ_n's and Ψ_n^*'s are zero unless n contains the i-th molecule of type s. And from (10) and (27) we obtain

$$\psi_s(i_s) = -\frac{1}{kT}\frac{\partial w_s(i_s)}{\partial x_{i_s}}, \tag{30}$$

$$\psi_{st}((i_s)(j_t)) = -\frac{1}{kT}\frac{\partial w_{st}((i_s)(j_t))}{\partial x_{i_s}} \tag{31}$$

And similarly we obtain

$$\psi_s^*(i_s) = -\frac{1}{kT}\frac{\partial w_s^*(i_s)}{\partial x_{i_s}} \tag{32}$$

$$\psi_{st}^*((i_s)(j_t)) = -\frac{1}{kT}\frac{\partial w_{st}^*((i)(j))}{\partial x_{i_s}} \quad . \tag{33}$$

The function $W_n\{n\}$ for three molecules approach the sum of the single molecule functions when all distances between molecules are large. And Kirkwood[2)6)] assumed that for small distances they are given to good approximation by the sum of the functions for all the pairs. This assumption is certainly not exact but used by Born, Green[3)] and Rodriguez[7)] And it is equivalent to the assumption that $w_n\{n\}$, $w_n^*\{n\}$, $\psi_n\{n\}$ and $\psi_n^*\{n\}$ are zero except for the case where the set n contains only one or two molecules.

If we introduce the Kirkwood approximation mentioned above in (22), and if the asterisks refer to the fluid or glassy system, for which $w_s^*(i_s)$ and $\psi_s^*(i_s)$ are zero for all s, we obtain the following integral equations:

$$\psi_s(i_s) = \sum_{r=1}^{\sigma} z_r \int_V \psi_{sr}^*((i_s)(k_r)) K((i_s);\ (i_s);\ (k_r))\, d(k_r), \tag{34}$$

$$\psi_{st}((i_s)(j_t)) = \psi_{st}^*((i_s)(j_t)) + \sum_{r=1}^{\sigma}\int_V \psi_{sr}^*((i_s)(k_r)) K((i_s)(j_t);\ (i_s);(k_r))\, d(k_r). \tag{35}$$

Here the kernels are given by

$$K((i_s);\ (i_s);(k_r)) = \frac{G_{sr}((i_s)(k_r))}{G_s(i_s)}, \tag{36}$$

and

$$K((i_s)(j_t);(i_s);(k_r)) = \frac{G_{str}((i_s)(j_t)(k_r))}{G_{st}((i_s)(j_t))} - \frac{G_{sr}((i_s)(k_r))}{G((i_s)}. \tag{37}$$

Inserting (36) in (34) and (37) in (35) and subtracting (34) from (35), we have

$$\psi_{st}((i_s)(j_t))-\psi_s(i_s)=\psi^*_{st}((i_s)(j_t))+\sum_{r=1}^{\sigma} z_r\int_V \psi^*_{sr}((i_s)(k_r))\frac{G_{str}((i_s)(j_t)(k_r))}{G_{st}((i_s)(j_t))}d(k_r). \tag{38}$$

(34) and (38) can be, according to (15), rewritten in the forms

$$\psi_s(i_s)=\sum_{r=1}^{\sigma}\frac{\rho_{\alpha r}(z_{\alpha r}-z_{\beta r})}{z_{\alpha r}}\int_V \psi^*_{sr}((i_s)(k_r))\frac{F_{sr}((i_s)(k_r))}{F_s(i_s)}d(k_r), \tag{39}$$

and

$$\psi_{st}((i_s)(j_t))-\psi_s(i_s)=\psi^*_{st}((i_s)(j_t)) + \sum_{r=1}^{\sigma}\frac{\rho_{\alpha r}(z_{\alpha r}-z_{\beta r})}{z_{\alpha r}}\int_V \psi^*_{sr}((i_s)(k_r))\frac{F_{str}((i_s)(j_t)(k_r))}{F_{st}((i_s)(j_t))}d(k_r). \tag{40}$$

It will be easily seen from (26) and (27) that the Kirkwood approximation can be expressed in the form

$$F_{str}((i_s)(j_t)(k_r))=\frac{F_{st}((i_s)(j_t))F_{tr}((j_t)(k_r))F_{rs}((k_r)(i_s))}{F_s(i_s)F_t(j_t)F_r(k_r)}. \tag{41}$$

Substitution of (41) into (40) leads to

$$\psi_{st}((i_s)(j_t))-\psi_s(i_s)=\psi^*_{st}((i_s)(j_t)) + \sum_{r=1}^{\sigma}\frac{\rho_{\alpha r}(z_{\alpha r}-z_{\beta r})}{z_{\alpha r}}\cdot\frac{1}{F_s(i_s)F_t(j_t)}\int_V \psi^*_{sr}((i_s)(k_r))\frac{F_{tr}((j_t)(k_r))F_{rs}((k_r)(i_s))}{F_r(k_r)}d(k_r). \tag{42}$$

§ 5. The Born-Green equation

Let us consider the case where the both values of the fugacity, z_α and z_β correspond to the gaseous, liquid or glassy states. Then $\psi_s(i_s)$ and $w_s(i_s)$ are zero. In this case (42) can be written in the form

$$\frac{\partial F_{st}((i_s)(j_t))}{\partial x_{ts}}+\frac{F_{st}((i_s)(j_t))}{kT}\frac{\partial w^*_{st}((i_s)(j_t))}{\partial x_{ts}} = \sum_{r=1}^{\sigma}\frac{\rho_{\alpha r}(z_{\alpha r}-z_{\beta r})}{z_{\alpha r}}F_{st}((i_s)(j_t))\int\frac{F_{rs}((k_r)(i_s))F_{ts}((j_t)(k_r))}{kT}\frac{\partial W^*_{rs}((k_r)(i_s))}{\partial x_{kr}}d(k^r), \tag{43}$$

where $w_{st}((i_s)(j_t))$ is assumed to be a function of (i_s) and (j_t) only through the distance between molecules i_s and j_t.

For the system composed of pure substance (43) becomes

$$\frac{\partial F_2((i)(j))}{\partial x_i}+\frac{F_2((i)(j))}{kT}\frac{\partial W_2^*((i)(j))}{\partial x_i}$$
$$=-\frac{\rho_\alpha(z_\alpha-z_\beta)}{z_\alpha}F_2((i)(j))\int_V\frac{F_2((i)(k))F_2((j)(k))}{kT}\frac{\partial W_2^*((i)(k))}{\partial x_k}d(k). \quad (44)$$

Here we denote F_{ss} and w_{ss} for a single component system by F_2 and w_2 respectively. If the asterisks refer to infinite dilution, $w_2((i)(j))$ is equal to $u(i, j)$, the potential between the pair of molecules i and j. Then (44) becomes

$$\frac{\partial F_2((i)(j))}{\partial x_i}+\frac{F_2((i)(j))}{kT}\cdot\frac{\partial u(i, j)}{\partial x_i}$$
$$=\rho F_2((i)(j))\int_V\frac{F_2((i)(j)F_2((j)(k))}{kT}\frac{\partial u(i, k)}{\partial x_k}d(k), \quad (45)$$

in which $\rho=\rho_\alpha-\rho_\beta$ and is equal to ρ_α if $\rho_\beta=0$. This integral equation was first derived by Born and Green[3]. Hence we conclude that (43) is the generalization of the Bron-Green equation, and a statistical-mechanical theory of fluid mixture will be evolved, based on the integral equation (43).

If ψ_n^*'s refer to infinite dilution and ψ_n's refer to solid state, we can write (39) in the form

$$\frac{\partial F_s(i_s)}{\partial x_{i_s}}=\sum_{r=1}^{\sigma}\rho_r\int_V\frac{F_{sr}((i_s)(k_r))}{kT}\frac{\partial u_{sr}(i_s, k_r)}{\partial x_{k_r}}d(k_r), \quad (46)$$

in which $u_{st}(i_s, j_t)$ is the potential energy of a pair of molecules of types s and t, i_s and j_t; and for single component system

$$\frac{\partial F_1(i)}{\partial x_i}=\rho\int\frac{F_2((i)(k))}{kT}\frac{\partial u(i, k)}{\partial x_k}d(k). \quad (47)$$

Eq. (47) resembles the equation used by Kirkwood and Monroe[8] in their theory of fusion, and is essentially equivalent to the equation used by Osawa and Satô[9] to discuss the multimolecular adsorption.

§ 6. The Kirkwood equation for pair distribution function

In the present section we assume that the asterisks refer to infinite dilution, and the functions $G_n^*(y)$ have the form

$$G_n^*(\{n\}, y)=\exp\left(-\frac{U_n(\{n\}, y)}{kT}\right), \quad (48)$$

where

$$U_n(\{n\}, y)=\sum_{s=1}^{\sigma}\sum_{t=1}^{\sigma}\sum_{l_s}^{n_s}\sum_{j_t}^{n_t}u_{st}(l_s, j_t)+(y-1)\sum_{t=1}^{\sigma}\sum_{jt}u_{st}(i_s, j_t), \quad (49)$$

the potential energy of the system being assumed to be expressible as a sum of

terms corresponding to the potentials of pairs of molecules. Since $U_n(\{n\}, 1)$ is equal to the potential energy of molecules n, $U_n\{n\}$, we have

$$G_n^*(\{n\}, 1)=\exp\left(-\frac{U_n\{n\}}{kT}\right), \tag{50}$$

which is equal to $G_n^*\{n\}$ for $z=0$. And from (33), we obtain

$$\psi_{st}^*((i_s)(j_t))=-\frac{u_{st}(i_s, j_t)}{kT}. \tag{51}$$

$\psi_{rp}^*((k_r)(l_p))$'s are zero unless either k_r or l_p is i_s. And in this case $\psi_n^*\{n\}$'s are zero unless n contains two and only two molecules. Then we obtain, from (15), (22) and (51),

$$\psi_{st}((i_s)(j_t))=-\frac{1}{kT}\Big[u_{st}(i_s, j_t)$$

$$+\sum_{r=1}^{\sigma}\rho_r\int_V u_{sr}(i_s, k_r)\left[\frac{F_{str}((i_s)(j_t)(k_r), y)}{F_{st}((i_s)(j_t), y)}-\frac{F_{sr}((i_s)(k_r), y)}{F_s((i_s), y)}\right]d(k_r) \tag{52}$$

by the similar procedure as used in the derivation of (34). According to (11) (15), (19) and (26), $\psi_{st}((i_s)(j_t))$ is given as

$$\psi_{st}((i_s)(j_t))=-\frac{1}{kT}\frac{\partial w_{st}((i_s)(j_t), y)}{\partial y}. \tag{53}$$

Inserting (53) in the left-hand side of (52), and replacing F_n by w_n, we have

$$\frac{\partial w_{st}((i)(j_t), y)}{\partial y}=u_{st}(i_s, j_t)$$

$$+\sum_{r}^{\sigma}\rho_r\int_V u_{st}(i_s, k_r)\left[\exp\left\{-\frac{W_{str}((i_s)(j_t)(k_r), y)-W_{st}((i_s)(j_t), y)}{kT}\right\}\right.$$

$$\left.-\exp\left\{-\frac{W_{sr}((i_s)(k_r), y)-W_s((i_s), y)}{kT}\right\}\right]d(k_r). \tag{54}$$

Here $w_n(\{n\}, y)$'s are defined by (15) and (26), if one replaces G_n by $G_n(y)$. Since $W_{str}((i_s)(j_t)(k_r), y)-W_{st}((i_s)(j_t), y)$ is the potential of average force upon k_r, when i_s and j_t held fixed and $W_{sr}((i_s)(k_r), y)-W((i_s), y)$ is the potential of average force upon k_r when i_s held fixed. Eq. (54) is the one which Kirkwood has already derived.

Here we will consider a fluid or glassy system for which $w_s((i_s), y)$ is zero at least for $y=1$. Bearing this in mind, expressing $W_n(\{n\}, y)$ in terms of $w_n(\{n\}, y)$ and using the Kirkwood approximation mentioned in § 4, we integrate (54) between zero and y with the result:

$$w_{st}((i_s)(j_t), y) = y u_{st}((i_s)(j_t))$$
$$+\sum_{r=1}^{\sigma} \rho_r \int_0^y \int u_{sr}(i_s, k_r) \exp\left\{-\frac{w_{sr}((i_s)(k_r), y)}{kT}\right\}\left[\exp\left\{-\frac{w_{tr}((j_t)(k_r))}{kT}\right\}-1\right] d(k_r)dy. \quad (55)$$

Here we assume that ρ_r and $w_{tr}((j)(k_r))$ for $i \neq j$, $t \neq s$, do not depend on y, since the variations of these quantities with y are of completely negligible order, $O(1/n_s)$. And for a single component system (55) becomes

$$w_2((i)(j), y) = y u_2 (i, j)$$
$$+\rho \int_0^y \int_V u_2(i, k) \exp\left\{-\frac{w_2((i)(k), y)}{kT}\right\}\left[\exp\left\{-\frac{w_2((i)(k))}{kT}\right\}-1\right] d(k)dy. \quad (56)$$

Eq.(56) has been utilized by Kirkwood, Monroe[9] and the present author[10] in order to calculate the pair distribution function of liquid argon from the interatomic force.

§ 7. The Kirkwood equation for single-molecule function

Let us consider a solid system, for which $\psi_s(i_s)$ and $w_s(i_s)$ are not zero. And, in this section, we will confine ourselves to the case of single component system to avoid complicated arguments. If $U_n(\{n\}, y)$ depends on y in such a manner that given by (47), and if the asterisks refer to infinite dilution, we obtain, from (22) and (23),

$$\psi_0 + \psi_1(i) = \rho \int_V u(i, k) \frac{F_2((i)(k), y)}{F_1((i), y)} d(k). \quad (57)$$

According to (15), (16) and (26), $\psi_1(i)$ is expressed by

$$\psi_0 + \psi_1(i) = \Psi_1(i) = \frac{V}{kT} \frac{\partial p(y)}{\partial y} + \frac{\partial \ln \rho}{\partial y} - \frac{1}{kT} \frac{\partial w_1(i)}{\partial y},$$

in which $p(y)$ is defined by

$$\exp \frac{p(y)V}{kT} = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int\int \cdots \int \exp\left\{-\frac{U_m(\{m\}, y)}{kT}\right\} d\{m\}. \quad (59)$$

We introduce (58) into (57) and integrate it between zero and y, obtaining

$$\frac{V\{p(y)-p(o)\}}{kT} + \ln \frac{\rho(y)}{\rho(o)} - \frac{w_1((i), y)}{kT}$$
$$= \int_0^y \int \rho u(i, k) \frac{F_2((i)(k), y)}{F_1((i), y)} d(k)dy, \quad (60)$$

in which the constant of integration is determined by the fact that $w_1((i), y)$ vanishes for $y=0$.

According to McMillan and Mayer[5], we have

$$\left(\frac{\partial p}{\partial z}\right)_{T,v} = kT\frac{\rho}{z}. \tag{61}$$

Differentiating (59) with respect to z, and making use of (61), we obtain

$$V\left\{\exp\frac{p(y)V}{kT}\right\}\frac{\rho}{z} = \sum_{m\geq 0}\frac{z^m}{m!}\int\int_V\cdots\int\exp\left\{-\frac{U_{m+1}(\{m\}+(i))}{kT}\right\}d\{m\}d(i) \tag{62}$$

Then we differentiate (62) with repect to y, obtaining

$$\frac{V}{kT}\left[\exp\frac{p(y)V}{kT}\right]\frac{\rho}{z}\frac{\partial[p(y)V+kT\ln\rho(y)]}{\partial y}$$

$$=\sum_{m\geq 0}\frac{z^m}{m!}\int\int_V\cdots\int\left[-\frac{1}{kT}\frac{\partial U_{m+1}(\{m\}+(i),y)}{\partial y}\right]\exp\left\{-\frac{U_{m+1}(\{m\}+(i),y)}{kT}\right\}$$
$$d\{m\}d(i). \tag{63}$$

And if we use (49), (63) can be rewritten in the form

$$\frac{V}{kT}\cdot\frac{\rho}{z}\left[\exp\frac{p(y)V}{kT}\right]\frac{\partial[p(y)V+kT\ln\rho(y)]}{\partial y}$$

$$=-\frac{z}{kT}\sum_{m\geq 0}\frac{z^m}{m!}\int\int_V\cdots\int u_{sr}(i_s,k_r)\exp\left\{-\frac{U_{m+2}(\{m\}+(i)+(k),y)}{kT}\right\}d\{m\}d(i)d(k). \tag{64}$$

Using (15), (17) and (48), we now obtain

$$\frac{V}{kT}\frac{\partial[p(y)V+kT\ln\rho(y)]}{\partial y} = -\frac{\rho}{kT}\int\int_V u(i,k)F_2((i)(k),y)d(i)d(k). \tag{65}$$

Now let us define $U(y)$ by

$$U(y) = \frac{\rho}{V}\int\int_V u(i,k)F_2((i)(k),y)d(i)d(k). \tag{66}$$

Then we obtain from (65)

$$V\{p(y)-p(o)\} + \ln\frac{\rho(y)}{\rho(o)} = -\frac{1}{kT}\int_0^1 U(y)\,dy. \tag{67}$$

$$-\frac{w_1((i),y)}{kT} = \int_0^y\int_V\rho u(i,k)\frac{F_2((i)(k),y)}{F_1((i).y)}d(k)dy + \frac{1}{kT}\int_0^y U(y)dy. \tag{68}$$

Here $\rho(1)-\rho(0)$ may be negligible compared with $\rho(1)$ or $\rho(0)$. Bearing the above mentioned in mind and using (26) and (66), we can rewrite (68) in the following form:

$$\ln\lambda F_1(i) = \int_V M((i)(k))F_1(k)d(k). \tag{69}$$

where

$$\ln \lambda = -\frac{\rho}{kT}\int_0^1\int_V u(i,k)F_2((i)(k),y)d(i)d(k)dy, \tag{70}$$

$$M((i)(k)) = -\frac{\rho}{kT}u(i,k)L((i)(k)), \tag{71}$$

$$L((i)(k)) = \int_0^1 g((i)(k),y)dy, \tag{72}$$

and

$$g((i)(k),y) = \frac{F_2((i)(k),y)}{F_1((i),y)F_1(k)} \tag{73}$$

The non-linear integral equation (69) has been used by Kirkwood and Monroe[8] in the theory of fusion.

§ 8. Conclusion

According to Mayer's general varitional method, we consider possible variations of the potential of average force. Then we obtain the Born-Green integral equation and the Kirkwood integral equation for the pair distribution function if the Kirkwood approximation is introduced.

As shown in § 5, this variational method shows that the generalization of the Born-Green equation to multicomponent systems and the one to the case where both of the fugacity sets for which two sets of the pair distribution functions are related are not zero are possible. This method will play an important rôle in the theory of liquid solutions. And this will be used to make clear the origin of the discrepancy between Mayer's statement and Rodriguez's one, as to the relation between the singularity in an isothermal and the equilibrium of liquid and gas.

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Theory of the Temperature Effect of Electronic Energy Bands in Crystals

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1. It is well-established both theoretically and experimentally for the crystal electrons that there exist the rather predominant interactions between the lattice vibrations and the electron in crystal, which play an important role in elucidating the temperature-dependent properties of the crystals. For example, the absorbed energy by crystal electrons from the incident radiation cannot usually be reemitted except for some of the specially treated crystals, i.e., crystalphosphors, coloured alkali halides etc., but may be transformed into the lattice vibration energy, thus resulting in the appearance of the characteristic absorption alone. From the same reasoning the mentioned interaction should be expected clearly to exhibit itself in the phenomena of the temperature-shift of electronic energy band in crystals, which will be experimentally observable, owing to the approximately continuous distributions within each of the energy band, at both ends of an energy band separated from each other by the forbidden region of finite breadth characteristic for both insulators and semiconductors. From the theoretical view point, since the temperature shifts of the electronic energy band are clearly ascribed to the reactive effect of the interactions with lattice vibrations of crystal electrons, the mentioned shifts of the edges and of energy band may be considered to be consisted of two kinds of terms, i.e., the "self-energy" term and the "radiative damping" one, similarly to the case of the electromagnetic interaction with free or bound electrons.[1] The former is found to give rise to the finite amount of energy shift due to the finite number of degrees of freedom of crystal lattice, while the latter to the level-broadening specified by the half-value breadth of the energy level, which states of affairs may be represented through the wave function $\psi(\boldsymbol{r}, t)$ of the crystal electrons in the foliowing manner;

$$\psi(\boldsymbol{r}, t) = \exp\{-i(E_0+\delta E)\, t/\hbar - \Gamma t\} \exp(i\boldsymbol{k}\boldsymbol{r}) u_{\boldsymbol{k}}(\boldsymbol{r}),$$

in which E_0 expresses the electronic energy without taking into account the interactions with lattice vibrations, δE the level shift of the "self energy" type, and $\hbar\Gamma'$ the half-value breadth due to the "radiative damping." Further, the wave number vector $\boldsymbol{k}$ specifies each of the electronic levels in crystal. Since the energy distribution of lattice quanta is well known to be governed by Planck's law at

thermodynamical equilibrium, both quantities of δE and $\hbar\Gamma'$ should be dependent, to a more or less amount, upon the temperature of crystal.

It is already observed[2),(3)] experimentally that in many crystalline insulators there is a displacement of absorption toward longer wave lengths with increasing temperature, which phenomena is simply manifested in many cases by the deepening of the colour of crystal. On the other hand, the temperature-shift of the absorption toward the red side has been theoretically discussed in detail by Radkowsky[4)] by taking into account the level broadening due to the radiative damping effect through the interaction with lattice vibrations, which effect seems to give a successful explanation for the mentioned shift in polar crystals since the thermal expansion effect is too small by a factor of at least four as shown by Möglich and Rompe.[3)] Recently the temperature-shift in non-polar crystals (Si) has been observed experimentally by Pearson and Bardeen[5)] through the detailed analysis of the changes with temperature of the electrical resistance and the Hall constant of the mentioned crystals, which results are shown easily, by a simple calculation, to become too large to be explained by the level broadening in Radkowsky's theory. In the present paper we shall show that the observed temperature-shift of the energy level in non-polar crystal may be successfully attributable to the "self-energy" effect through the interactions with lattice vibrations and, further, the level broadening in polar crystals becomes of comparable order of magnitudes with the "self-energy" effect, thus the total contribution of both effects being responsible for the level shift of the mentioned crystals.

2. Hamiltonian of an electron under the combined influence of both the electrostatic field with crystal period and the lattice vibration may be written as follows.

For the semimetal (non-polar monoatomic crystal),[6)] we have

$$H=\underbrace{-\frac{\hbar^2}{2m}\Delta+V(\boldsymbol{r})}_{H_e}+\underbrace{\frac{1}{2}\sum_{t,\boldsymbol{w}}\{p_t^2(\boldsymbol{w})+4\pi^2\nu_t^2(\boldsymbol{w})q_t^2(\boldsymbol{w})\}}_{H_L}-\underbrace{\boldsymbol{R}\,\mathrm{grad}\,V(\boldsymbol{r})}_{\delta H}, \quad (1)$$

$$\boldsymbol{R}=\frac{1}{\sqrt{MN}}\sum_{t,\boldsymbol{w}}\boldsymbol{\xi}_t\{a_t(\boldsymbol{w})\exp(i\boldsymbol{w}\boldsymbol{r})+a_t^*(\boldsymbol{w})\exp(-i\boldsymbol{w}\boldsymbol{r})\}, \quad (2)$$

in which H_e = Hamiltonian of an electron in the crystal field,

H_L = " of the lattice vibrations,

δH = " of the interaction with the lattice vibrations of an electron,

$t(1,2,3)$ = Suffix specifying each of three modes of vibrations with different polarization,

$\boldsymbol{\xi}_t$ = Unit vector of polarization of elastic waves denoted by t,

p_t, a_t = Canonical momentum and coordinate of a normal mode of vibration corresponding to the wave number vector $\boldsymbol{w}$ and polarization t,
$\nu_t(\boldsymbol{w})$ = Frequency of a normal mode of vibration belonging to $\boldsymbol{w}$ and t,
M = Mass of an atom situated at each lattice point,
N = Number of atoms involved in the crystal.

Further, $a_t(\boldsymbol{w})$ and $a_t^*(\boldsymbol{w})$ express, as usual, the operators responsible for the absorption and the emission of a phonon, whose non-vanishing matrix elements are

$$\left.\begin{aligned}(N_t(\boldsymbol{w})+1|a_t^*(\boldsymbol{w})|N_t(\boldsymbol{w}))&=\sqrt{\frac{\hbar}{4\pi\nu_t(\boldsymbol{w})}}(N_t(\boldsymbol{w})+1),\\(N_t(\boldsymbol{w})-1|a_t(\boldsymbol{w})|N_t(\boldsymbol{w}))&=\sqrt{\frac{\hbar}{4\pi\nu_t(\boldsymbol{w})}}N_t(\boldsymbol{w}).\end{aligned}\right\}\tag{3}$$

$N_t(\boldsymbol{w})$ represents a phonon-number, the average value of which is given, at thermodynamical equilibrium, by Planck's law

$$N_t(\boldsymbol{w})=\{\exp(h\nu_t(\boldsymbol{w})/k_0T)-1\}^{-1},\tag{4}$$

k_0 and T being Boltzmann constant and absolute temperature.

For the ionic crystal (diatomic polar crystal),[7][8] as shown by Born and v. Karman, the normal modes of the elastic vibrations of the mentioned crystals can be divided up into two branches, the acoustic and the optical one. The number of normal modes in each branch is $3N$ when $2N$ is the number of ions. The oscillations of the optical branch are those in which two neighbouring ions of opposite sign vibrate in opposite directions, while, in the acoustical branch, neighbouring ions of opposite signs have almost the same displacement. Since the optical branch gives much the greater polarization of the lattice, it gives a much stronger interaction with the electron. So we shall be allowed, in the following, to take into account only the optical vibration in the mutual interaction with crystal electrons except for the case of extremely low temperature where the contribution of the acoustical branch to the interaction energy may be rather appreciable, owing to the increasing phonon number, in comparison with that of the optical branch.

Now we have

$$H=H_e+\underbrace{\sum_{\boldsymbol{w}}\left(\frac{Y_w^2}{2M}+2\pi^2\nu^2MX_w^2\right)}_{H_L}+\underbrace{\frac{2\pi e^2}{a^3(2N)^{1/2}}\sum_{\boldsymbol{w}}\frac{1}{w}\left\{X_w\sin(\boldsymbol{wr})+\frac{X_wY_w}{2\pi\nu M}\cos(\boldsymbol{wr})\right\}}_{\delta H},$$

in which the real canonical coordinate and momentum X_w, Y_w are taken to designate each normal mode of lattice vibration of the optical branch, a the lattice constant and ν the frequency. Since the vibrations of optical branch have all nearly the same frequency ν, it will involve only a small error if we give the Reststrahlen frequency ν to all waves.

The matrix elements of X_w and Y responsible for a phonon-absorption and -emission may be given by

$$\left.\begin{aligned}(N_w \pm 1|X_w|N_w) &= \mp i\sqrt{\frac{\hbar}{4\pi M\nu}\left(N_w+\frac{1}{0}\right)},\\ (N_w \pm 1|Y_w|N_w) &= \sqrt{\frac{\hbar}{4\pi M\nu}\left(N_w+\frac{1}{0}\right)}.\end{aligned}\right\} \tag{6}$$

3. Now we shall evaluate the correction to the electronic energy in crystal owing to the mutual interaction with electrons of the lattice vibrations described by (1) and (5). Since the first order energy correction is easily shown to vanish due to (3) and (6), we have to manipulate the second order correction of the electronic energy, i.e., the correction of the "self-energy" type, which shall be found to become of a finite magnitude owing to a finite degree of freedom of crystal lattice in contradiction to the case of radiation field (infinite degree of freedom).

Thus, it follows, for the second order energy correction δE,

$$\delta E = \sum_{\boldsymbol{k}',w',t} \frac{|(\boldsymbol{k}, N_t(w)|\delta H|\boldsymbol{k}', N_t'(w')|^2}{E_0(\boldsymbol{k}, N_t(w)) - E_0(\boldsymbol{k}', N_t'(w'))}, \tag{7}$$

in which $E_0(\boldsymbol{k}, N_t(w))$ represents the energy of the system H_e+H_L without the mutual interaction and $\boldsymbol{k}$ the wave number vector of a crystal electron specifying each of the electronic states.

Now, for brevity, we shall confine ourselves to a simple cubic lattice, which model will be sufficient for our present calculation, since our results obtained will easily be extended to the more complex lattice structure. In such case, we have, as usual,

$$E_u = \hbar^2 k^2/2m_1^*, \tag{8}$$

for the energy near the bottom of the upper band, and

$$E_l = -E_0 - \frac{\hbar^2}{2m_2^*}\left(\frac{\pi}{a}\boldsymbol{n}-\boldsymbol{k}\right)^2, \tag{9}$$

for that near the top of the lower band respectively, in which m_1^* and m_2^* express the effective masses, $\boldsymbol{n}$ the unit vector whose components are integral numbers, a the lattice constant, and E_0 the energy gap between the bands 1 and 2 without the energy correction under consideration, the bottom of the upper band being taken as an energy origin. The energy expression (9) may easily be derived from the energy formula obtained in Bloch's tightly bound electron approximation.[6] The eigenfunctions corresponding to (8) and (9) may be written by

$$\psi_{\boldsymbol{k}}^{(1)}(\boldsymbol{r}) = N^{-1/3}\exp(i\boldsymbol{k}\boldsymbol{r})u_{\boldsymbol{k}}^{(1)}(\boldsymbol{r}), \tag{10}$$

and

$$\psi_{\boldsymbol{k}}^{(2)}(\boldsymbol{r})=N^{-1/3}\exp(i\boldsymbol{k}\boldsymbol{r})u_{\boldsymbol{k}}^{(2)}(\boldsymbol{r}), \tag{11}$$

respectively, in which $u_{\boldsymbol{k}}^{(1)}(\boldsymbol{r})$ and $u_{\boldsymbol{k}}^{(2)}(\boldsymbol{r})$ are periodic functions with a crystal period and may be supposed to be nearly independent of $\boldsymbol{k}$ in the energy regions under consideration.

Semimetal (Nonpolar crystal) :

For the non-vanishing matrix elements of δH, we have, as already shown in the theory of electrical conductivity,[6]

$$\left.\begin{aligned}(\boldsymbol{k},N_t(\boldsymbol{w})|\delta H|\boldsymbol{k}-\boldsymbol{w},N_t(\boldsymbol{w})+1)&=\sqrt{\frac{\hbar}{MN4\pi\nu_t(\boldsymbol{w})}(N_t(\boldsymbol{w})+1)}\cdot\frac{-2i}{3}\boldsymbol{w}C_1,\\(\boldsymbol{k},N_t(\boldsymbol{w})|\delta H|\boldsymbol{k}+\boldsymbol{w},N_t(\boldsymbol{w})-1)&=\sqrt{\frac{\hbar}{MN4\pi\nu_t(\boldsymbol{w})}N_t(\boldsymbol{w})}\cdot\frac{2i}{3}\boldsymbol{w}C_1,\end{aligned}\right\} \tag{12}$$

respectively, in which the longitudinal waves ($t=1$) of lattice vibrations only are shown to make contributions to the scattering matrix elements approximately. For the case of the electron or holes near the top of band 2, the similar expression for the scattering matrix to (12) will be valid except for the replacement of C_1 by C_2, corresponding to (10) and (11), where the average kinetic energies C_1 and C_2 are given by

$$C_i=\frac{3\hbar^2}{2m_i^*}\int_{\tau_0}\left|\frac{\partial u_{\boldsymbol{k}}^{(t)}(\boldsymbol{r})}{\partial s}\right|^2 d\tau, \quad (i=1,2) \tag{13}$$

in which τ_0 and s denote the volume of elementary cell in crystal and the displacement along the direction of polarization of longitudinal wave respectively.

Ionic crystal (Diatomic polar crystal) :

$$(\boldsymbol{k},N(\boldsymbol{w})|\delta H|\boldsymbol{k}\mp\boldsymbol{w},N(\boldsymbol{w})\pm 1)=\frac{2\pi e^2}{a^3(2N)^{1/2}}\sqrt{\frac{\hbar}{4\pi M\nu}\left(N(\boldsymbol{w})+\frac{1}{0}\cdot\right)}\frac{1}{\boldsymbol{w}}C_i', \tag{14}$$

where C_i' represents the constant quantity being nearly equal to one, and $i=1,2$ specifies each of the bands 1 and 2 respectively.

Taking into account (12) and (14), the energy correction δE in (7) is easily seen to take the following simplified form,

$$\delta E=\sum_{\boldsymbol{w}}\left\{\frac{|(\boldsymbol{k}-\boldsymbol{w},N_1(\boldsymbol{w})+1|\delta H|\boldsymbol{k},N_1(\boldsymbol{w}))|^2}{E_0(\boldsymbol{k},N_1(\boldsymbol{w}))-E_0(\boldsymbol{k}-\boldsymbol{w},N_1(\boldsymbol{w})+1)}+\frac{|(\boldsymbol{k}+\boldsymbol{w},N_1(\boldsymbol{w})-1|\delta H|\boldsymbol{k},N_1(\boldsymbol{w}))|^2}{E_0(\boldsymbol{k},N_1(\boldsymbol{w}))-E_0(\boldsymbol{k}+\boldsymbol{w},N_1(\boldsymbol{w})-1)}\right\}, \tag{15}$$

in which the summation over $\boldsymbol{w}$ may be evaluated approximately, as usual, by replacing by the following integration,

$$\sum_{\boldsymbol{w}}\cdots\longrightarrow\frac{Na^3}{(2\pi)^3}\int_{-\pi/a}^{\pi/a}dw_1\int_{-\pi/a}^{\pi/a}dw_2\int_{-\pi/a}^{\pi/a}dw_3\cdots$$

$$=\frac{Na^3}{(2\pi)^3}\int_0^{w_0}w^2dw\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\cdots, \tag{16}$$

$$w_0=\sqrt[3]{6\pi^2}/a,$$

taking into account the range of values of each component of $\boldsymbol{w}$, namely,

$$-\pi/a\leqq w_1, w_2, w_3\leqq\pi/a. \tag{17}$$

Thus, it follows, for the case of semimetal (nonpolar crystal),

$$\delta E_u(\boldsymbol{k}=0)=\frac{2\hbar C_1^2}{9MNc}\frac{Na^3}{(2\pi)^3}\int w^2dw\sin\theta d\theta d\varphi\left\{-\frac{N_1(\boldsymbol{w})+1}{a_1w+\gamma}+\frac{N_1(\boldsymbol{w})}{-a_1w+\gamma}\right\}, \tag{18}$$

for the bottom of the upper band and

$$\delta E_l(\boldsymbol{k}=\frac{\pi}{a}\boldsymbol{n})=-\frac{2\hbar C_2^2}{9MNc}\frac{Na^3}{(2\pi)^3}\int w^2dw\sin\theta d\theta d\varphi\left\{-\frac{N_1(\boldsymbol{w})+1}{a_2w+\gamma}+\frac{N_1(\boldsymbol{w})}{-a_2w+\gamma}\right\}, \tag{18}'$$

for the top of the lower band respectively, where c expresses the sound velocity, $a_i=\hbar^2/2m_i^*$ $(i=1,2)$ and $\gamma=\hbar c$.

$$T\gg\theta;\qquad N_1(\boldsymbol{w})\simeq\frac{k_0T}{h\nu}=\frac{k_0T}{\gamma w}. \tag{4'}$$

Then, we get

$$\delta E_u(\boldsymbol{k}=0)=-\frac{1\hbar a^3C_1^2}{9\pi^2Mc}\left\{\frac{1}{a_1}\left[\frac{w_0^2}{2}-\frac{\gamma}{a_1}w_0+\left(\frac{\gamma}{a_1}\right)^2\log\left(\frac{a_1}{\gamma}w_0+1\right)\right]\right.$$

$$\left.+\frac{1}{a_1\gamma}\left[2w_0+\frac{\gamma}{a_1}\log\frac{a_1w_0-\gamma}{a_1w_0+\gamma}\right]k_0T\right\}, \tag{19}$$

$$\delta E_l\left(\boldsymbol{k}=\frac{\pi}{a}\boldsymbol{n}\right)=\frac{1\hbar a^3C_2^2}{9\pi^2Mc}\left\{\frac{1}{a_2}\left[\frac{w_0^2}{2}+\frac{\gamma}{a_2}w_0+\left(\frac{\gamma}{a_2}\right)^2\log\left(\frac{a_2}{\gamma}w_0-1\right)\right]\right.$$

$$\left.+\frac{1}{a_2\gamma}\left[2w_0+\frac{\gamma}{a_2}\log\frac{a_2w_0-\gamma}{a_1w_0+\gamma}\right]k_0T\right\}. \tag{19}'$$

For the breadth of the forbidden region separating the bands 1 and 2, therefore, we obtain

$$\Delta E=\delta E_u(\boldsymbol{k}=0)-\{-E_0+\delta E_l(\boldsymbol{k}=\pi/a\cdot\boldsymbol{n})\}=E_G-\beta T, \tag{20}$$

where

$$E_G=E_0+\frac{1\hbar a^3}{9\pi^2Mc}\left\{\frac{C_1^2}{a_1}\left[\frac{w_0^2}{2}-\frac{\gamma}{a_1}w_0+\left(\frac{\gamma}{a_1}\right)\log\left(\frac{a_1}{\gamma}w_0+1\right)\right]\right.$$

$$+\frac{C_2^2}{a_2}\left[\frac{w_0^2}{2}+\frac{\gamma}{a_2}w_0+\left(\frac{\gamma}{a_2}\right)^2\log\left(\frac{a_2}{\gamma}w_0-1\right)\right]\Bigg\}, \tag{21}$$

$$\beta=\frac{1\hbar k_0 a^3}{9\pi^2 Mc\gamma}\left\{\frac{C_1^2}{a_1}\left[2w_0+\frac{\gamma}{a_1}\log\frac{a_1w_0-\gamma}{a_1w_0+\gamma}\right]+\frac{C_2^2}{a_2}\left[2w_0+\frac{\gamma}{a_2}\log\frac{a_2w_0-\gamma}{a_2w_0+\gamma}\right]\right\}. \tag{22}$$

The energy correction independent of T in (21) is due to the interaction with zero point vibrations of lattice, and, further, each term of (21) cannot be measured separately but only as a whole (E_G) as easily seen,

$$T \ll \theta, \qquad N_1(w) \cong \exp(-\hbar cw/k_0T). \tag{4}''$$

Substituting (4)'' into (18) and (18)', we obtain, after a similar process of manipulation to the case of $T \gg \theta$,

$$\Delta E=E_G-\frac{4a^3k_0^2}{9\pi^2M\hbar^3c^3}\Bigg\{m_1{}^*C_1^2\Bigg\{T^2-\exp\left(-\frac{\hbar w_0c}{k_0T}\right)\cdot\left(\frac{\hbar w_0c}{k_0}T+T^2\right)$$

$$+\frac{2m_1{}^{*2}c^4}{k_0^2}\exp\left(\frac{2m_1{}^*c^2}{k_0T}\right)\left[-E_i\left(-\frac{2m_1{}^*c^2}{k_0T}\right)+E_i\left(-\frac{\hbar w_0c}{k_0T}-\frac{2m_1{}^*c^2}{k_0T}\right)\right]$$

$$+\frac{2m_1{}^{*2}c^4}{k_0^2}\exp\left(-\frac{2m_1{}^*c^2}{k_0T}\right)\left[-\bar{E}_i\left(\frac{2m_1{}^*c^2}{k_0T}\right)+E_i\left(-\frac{\hbar w_0c}{k_0T}+\frac{2m_1{}^*c^2}{k_0T}\right)\right]\Bigg\}$$

$$+m_2C_2^2\Bigg\{T^2-\exp\left(-\frac{\hbar w_0c}{k_0T}\right)\cdot\left(\frac{\hbar w_0c}{k_0}T+T^2\right)$$

$$+\frac{2m_2{}^{*2}c^4}{k_0^2}\exp\left(\frac{2m_2{}^*c^2}{k_0T}\right)\left[-E_i\left(-\frac{2m_2{}^*c^2}{k_0T}\right)+E_i\left(-\frac{\hbar w_0c}{k_0T}-\frac{2m_2{}^*c^2}{k_0T}\right)\right]$$

$$+\frac{2m_2{}^{*2}c^2}{k_0^2}\exp\left(-\frac{2m_2{}^*c^2}{k_0T}\right)\left[-\bar{E}_i\left(\frac{2m_2{}^*c^2}{k_0T}\right)+E_i\left(-\frac{\hbar w_0c}{k_0T}+\frac{2m_2{}^*c^2}{k_0T}\right)\right]\Bigg\}\Bigg\}$$

$$\left(E_i(\delta)=\int_{\infty}^{-\delta}e^{-t}/t\cdot dt\right)$$

$$\cong E_G-\frac{4a^3k_0^2}{9\pi^2M\hbar^3c^3}(m_1{}^*C_1^2+m_2{}^*C_2^2)\left[T^2-\exp\left(-\frac{\hbar w_0c}{k_0T}\right)\cdot\left(\frac{\hbar w_0c}{k_0}T+T^2,\right)\right], \tag{22}'$$

since the terms involving E_i may be considered to become negligibly small in comparison with the remaining terms especially except for very low temperatures, as shown numerically.

For the case of the ionic (diatomic polar) crystal, we have, after performing the similar calculation to the above case of semimetal,

$$T \gg \theta;$$

$$\Delta E=E'_G-\beta'T, \tag{23}$$

where

$$E_G'=E_0-\frac{e^4\hbar}{4\pi a^3 M\nu}\left\{\frac{C_1'^2\tan^{-1}(\sqrt{a_1/h\nu}\cdot w_0)}{\sqrt{a_1 h\nu}}+\frac{C_2'^2}{\sqrt{a_2 h\nu}}\log\frac{\sqrt{a_2/h\nu}\cdot w_0-1}{\sqrt{a_2/h\nu}\cdot w_0+1}\right\}, \tag{24}$$

and

$$\beta'=\frac{e^4}{8\pi^2 a^3 M\nu^2}\left\{\left[\frac{\tan^{-1}(\sqrt{a_1/h\nu}\cdot w_0)}{\sqrt{a_1 h\nu}}+\frac{1}{2\sqrt{a_1 h\nu}}\log\frac{\sqrt{a_1/h\nu}\cdot w_0-1}{\sqrt{a_1/h\nu}\cdot w_0+1}\right]C_1'^2\right.$$
$$\left.+\left[\frac{\tan^{-1}(\sqrt{a_2/h\nu}\cdot w_0)}{\sqrt{a_2 h\nu}}+\frac{1}{2\sqrt{a_2 h\nu}}\log\frac{\sqrt{a_2/h\nu}\cdot w_0-1}{\sqrt{a_2/h\nu}\cdot w_0+1}\right]C_2'^2\right\}. \tag{25}$$

$T\ll\theta$;

$$\Delta E=\dot{E}_G'-\frac{e^4\hbar}{4\pi a^3 M\nu}\left\{\left[\frac{\tan^{-1}(\sqrt{a_1/h\nu}\cdot w_0)}{\sqrt{a_1 h\nu}}+\frac{1}{2\sqrt{a_1 h\nu}}\log\frac{\sqrt{a_1/h\nu}\cdot w_0-1}{\sqrt{a_1/h\nu}\cdot w_0+1}\right]C_1'^2\right.$$
$$\left.+\left[\frac{\tan^{-1}(\sqrt{a_2/h\nu}\cdot w_0)}{\sqrt{a_2 h\nu}}+\frac{1}{2\sqrt{a_2 h\nu}}\log\frac{\sqrt{a_2/h\nu}\cdot w_0-1}{\sqrt{a_2/h\nu}\cdot w_0+1}\right]C_2'^2\right\}$$
$$\times\exp\left(-\frac{h\nu}{k_0 T}\right). \tag{26}$$

4. Generally speaking, the combined effect of both self-energy-and radiation-damping types will give rise to the observed temperature-shift at the edges of the energy bands, but, for the real crystalline insulators and semiconductors, one or the other of the mentioned effects may be considered to predominate. In order to see such situation and, further, to proceed the comparison with the experimental results, we shall carry out the numerical estimates of ΔE for the cases of NaCl and Si as follows.

NaCl: $a=2.81\times10^{-8}$ cm,

$M=M_{Na}M_{Cl}/(M_{Na}+M_{Cl})=14.0\times$ proton mass,

$\nu=k_0\theta/h=0.583\times10^{13}$ $(\theta=280^\circ K)$,

$a_1=6.05\times10^{-28}/\varepsilon_1$, $m_1^*=\varepsilon_1 m$ (m=electron mass),

$a_2=6.05\times10^{-28}/\varepsilon_2$, $m^*_2=\varepsilon_2 m$,

$\gamma=3.86\times10^{-14}$,

$C_1'\sim C_2'\sim1$.

$$\Delta E=E_0-0.303(\sqrt{\varepsilon_1}-0.037\varepsilon_2)$$
$$-10.8\times10^{-4}(\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2})\left(1-0.037\frac{\varepsilon_1+\varepsilon_2}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\right)T\ (\mathrm{eV}). \tag{27}$$

Since we have at present no reliable data regarding ε_1 and ε_2, the values of $\varepsilon_1=\varepsilon_2=1$ shall be tentatively adopted in the following evaluation.

Then, we have

$$\Delta E = E_0 - 0.291 - 2.08 \times 10^{-3} T \quad \text{(eV)}, \tag{28}$$

the temperature coefficient of which is found to become of a comparable order of magnitude with that of the temperature-broadening due to the radiative damping effect as shown in Radkowsky's theory[4] and, therefore, the self-energy correction here computed should be taken into consideration, together with the level-broadening one, for the quantitative comparison with the experiments.

Si : $a = 2.7 \times 10^{-8}$cm

$M = 28$ proton mass $(= 46.8 \times 10^{-24}\text{gr})$

As for C_1^2, Seitz's calculated value[9] is 104 $(\text{eV})^2$ at $T = 295°K$, which leads to the electron mobility of 243 cm^2/volt sec, while, the experimental value is 300 cm^2/volt sec,[9] which gives rise to $C_1^2 = 84.2$ $(\text{eV})^2$ according to the well known relation

$$\mu_1 = \frac{3\sqrt{2\pi} e\hbar c M n_0}{2(k_0 T)^{3/2} (m_1^*)^{5/2} C_1^2} \cdot \qquad (n_0 = \text{the density of unit cells in the lattice}) \tag{29}$$

In the numerical evaluation of δE, the mentioned two values for C_1^2 have been taken respectively. Regarding the value of C_2^2 (average kinetic energy of the hole at the top of the lower band), we have made use of the observed value $\mu_1/\mu_2 = 3$ and the effective mass ratio mentioned below (Pearson and Bardeen)[5] together with the relation of

$$\frac{\mu_1}{\mu_2} = \frac{C_2^2}{C_1^2} \left(\frac{m_2^*}{m_1^*} \right)^{5/2}. \tag{30}$$

Pearson and Bardeen's experiment[5] shows $\varepsilon_2 \fallingdotseq 1$ and $\varepsilon_1 \sim 0.33$ or 0.65 according to whether we may estimate them by the observed depth of the donor level or by the mobility deduced from the changes with temperature of both Hall constant and electrical conductivity, but $\varepsilon_1 \sim 0.65$ seems to be a more plausible one. The sound velocity in Si has not yet been measured directly as far as we know, and, therefore, we have estimated it by the use of one or the other of the following relations,

$$\frac{1}{\tau} = \frac{4 m_1^* k_m C_1^2 k_0 T}{9\pi \hbar^3 \rho c^2}, \qquad (k_m = \sqrt[3]{6\pi^2}/a,\ \rho = \text{density}) \tag{31}$$

or

$$c = \frac{k_0 \theta}{h} \left(\frac{4\pi V}{3N} \right)^{1/3}, \qquad (V = \text{volume of the crystal}), \tag{32}$$

together with the calculated value for c of diamond (Seitz).[9] The results are

from (31)

$\varepsilon_1 = 1$: $c = 10800$ m/sec.

$\varepsilon_1 = 0.65$: $c = 8720$ m/sec.

and

$$c = 8900 \text{ m/sec.} \quad \text{from (32)}.$$

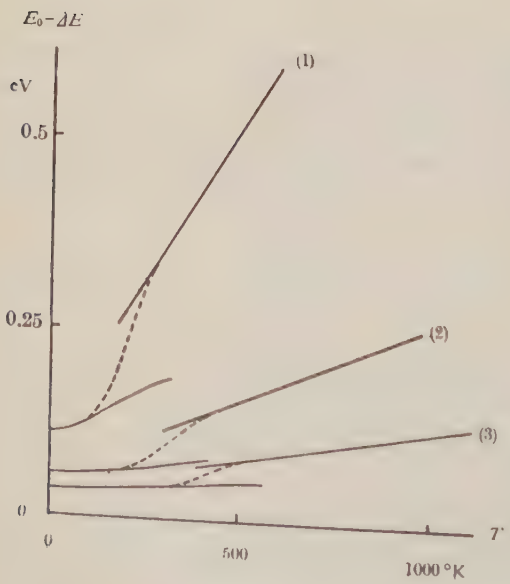
Taking into account the uncertainties involved in the estimated values of c mentioned above, we have attempted, for the numerical evaluation of ΔE, to take the values of $c = 0.5 \times 10^6$ cm/sec, 1.0×10^6 cm/sec and 1.5×10^6 cm/sec respectively.

Substituting the mentioned numerical values suitable for Si into (20) ~ (22), except for c, ε_1, ε_2 C_1 and C_2, we obtain

Table 1

	ε_1	C_2^2/C_1^2	c $\times 10^6$cm/sec	$C_1^2=104(eV)^2$ α	β	$C_1^2=84.2(eV)^2$ α	β
$\frac{C_2^2}{C_1^2}=\frac{\mu_1\varepsilon^{5/2}}{\mu_2\varepsilon^{5/2}}$,	1	3	0.5	0.113	8.0×10^{-4}	0.092	6.5×10^{-4}
			1.0	57	2.1	46	1.7
			1.5	38	0.9	31	0.75
$\frac{\mu_1}{\mu_2}=3.$	0.65	1.02	0.5	0.043	3.4×10^{-4}	0.038	2.8×10^{-4}
			1.0	24	0.25	19	0.70
			1.5	16	0.38	13	0.31
(Pearson and Bardeen)	0.33	0.189	0.5	0.015	1.1×10^{-4}	0.012	0.85×10^{-4}
			1.0	8	0.27	6	0.22
			1.5	5	0.12	4	0.10

$$\Delta E = E_0 - \alpha - \beta T \quad ev,$$
$$m_1^* = \varepsilon_1 m \; ; \quad m_2^* = \varepsilon_2 m \quad (\varepsilon_2 = 1)$$



$\varepsilon_1=\varepsilon_2=1$; $C_2^2/C_1^2=3$, $C_1^2=104(ev)^2$.

(1) $c=0.5\times10^6$ cm/sec ;

(2) $c=1.0\times10^6$ cm/sec ;

(3) $c=1.5\times10^6$ cm/sec.

Fig. 1.

$$\Delta E = E_0 - 1.35\times 10^{2}\frac{\varepsilon_1 C_1^2}{c}\left\{1+\frac{\varepsilon_2 C_2^2}{\varepsilon_1 C_1^2}-2.4\times 10^{-8}\varepsilon_1 c\left(1-\frac{\varepsilon_2^2 C_2^2}{\varepsilon_1^2 C_1^2}\right)\right\}$$

$$-4.93\times 10^{5}\frac{\varepsilon_1 C_1^2}{c^2}\left(1+\frac{\varepsilon_2 C_2^2}{\varepsilon_1 C_1^2}\right)T\ (ev),\ (C_i \text{ in } (cv),\ c \text{ cm/sec})$$

$$= E_0 - a - \beta T\ (ev), \tag{33}$$

and the numerical data of ΔE for each of the appropriate values of the quantities involved in (33) are shown in Table 1. The general trend of ΔE with temperatures is described in Fig. 1, including the low temperature region according to (26).

From the detailed analysis of the changes with temperature of the concentrations of both electrons in the upper band and holes in the lower band, Pearson and Bardeen[5] have observed, for the temperature coefficient β of ΔE, to become of 3×10^{-4} ev/degree for Si in the higher temperature range, which value is in fair agreement with our theoretical one for the suitable values of the physical quantities, i.e., $\varepsilon_1 \sim 0.65$, $c \sim 05.\times 10^6$ cm/sec, or $\varepsilon_1 \sim 1$, $c \sim 0.8\times 10^6$ cm/sec, as seen in Table 1. The contribution from the level broadening discussed by Radkowsky is easily seen to become negligibly small, for the case of Si, in comparison with that from the energy correction mentioned above.

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One of the authors expresses his hearty thanks to Dr. J. Bardeen for his private communication in which the elegant method of checking of our theory has been reported.

Note added in proof : After sending our manuscript of the present paper to the editor of Progress of Theoretical Physics, Fan's letter in Phys. Rev. 78, 1950, 808 has come to our notice in which the temperature-shift of the energy gap in monoatomic semiconductors has been worked on a quite similar line as ours and, further, the observed shifts with temperatures have been analyzed theoretically. Our criticism of Fan's theoretical analysis together with our calculated shifs for Ge and diamond shall be published soon in the forthcoming issue of the journal.

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Cosmic-Ray Underground, III

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§ 1. Introduction

Investigations concerning the cosmic-ray phenomena underground provide important clues to infer the nature of the cosmic-ray mesons of moderately high energies.

The directional intensity and the temperature effect of cosmic-ray underground are computed numerically in order to give further supports to the theory of Hayakawa and Tomonaga[1] which was proposed to explain the well-known bend in the intensity *vs.* depth curve of the cosmic-ray underground on the basis of the fact that π-meson decays into μ-meson and the latter, having no strong interaction with matter except an electromagnetic one, can penetrate far underground.

The result of the computation of the directional intensity shows fairly good agreement with the experiments of Greisen[2] and thus, though the calculated temperature effect is not yet quantitatively in good accord with the experimental data of Barnóthy and Forró[3][4], we may conclude that the rays existing deep in the ground consist of μ-mesons and they lose their energy according to the equation of Hayakawa and Tomonaga[1] which took only electromagnetic processes of μ-mesons into account, and consequently that the quantum electrodynamics proves its validity up to the high energy of some 10^{13} eV or more.

§ 2. The directional intensity

As was stated in I, most of soft π-mesons which were produced by the primary rays high in the atmosphere decay into μ-mesons before reaching the sea-level and hard π-mesons that arrive sea-level are immediately absorbed after penetrating into the ground, and μ-mesons play the leading rôle in the ground. The directional intensities at various depths may be obtained, following the same method as of I, by means of the energy-range relation of μ-mesons in the earth and the integral energy spectrum of obliquely incident μ-mesons on sea-level. Taking the absorption of π-mesons in the atmosphere into account the diffusion equation of π-mesons for the atmospheric depth l and the zenith angle of incidence θ is obtained from Eq. (19), I, by the substitution of $l \to l/\cos\theta$. And this results in the transformations of

$$\Lambda \to \Lambda \cdot \cos\theta, \tag{1}$$

and

$$B \to B/\cos\theta \tag{1'}$$

for the results obtained in I. Explicitly, we have next equation:

$$\frac{\partial g(E_0, l, \theta)}{\partial l/\cos\theta} = -\frac{B/\cos\theta}{E_0 l/\cos\theta} \cdot g(E_0, l, \theta) - \frac{1}{\Lambda} \cdot g(E_0, l, \theta) + g_0(E_0, l, \theta). \tag{2}$$

Here $g(E_0, l, \theta)$ means the differential energy spectrum in question, and $1/\Lambda$ the absorption coefficient of π-mesons for which we assume the same value as for nucleons: $\Lambda = 125 \mathrm{gcm}^{-2}$. And

$$B = (\mu_\pi c/\tau_\pi)(l/\rho) = 3.4 \times 10^2 \text{ BeV},$$

where μ_π, τ_π are the mass and the mean life-time of the π-meson respectively and ρ means the average density of air. The numerical value of B is derived by putting $\mu_\pi = 286m$ (m: electron mass), $\tau_\pi = 1 \times 10^{-8}$ sec.

By adopting for the source function, g_0, of π-mesons

$$g_0(E_0, l, \theta) = \text{const.}\ (E_0^{-2.8} - 2.2E_0^{-3.5}) \exp(-(l/\cos\theta)/\Lambda),$$

determined by Hayakawa and Nishimura[5], Eq. (1) can be solved and we have

$$g(E_0, l, \theta) = \text{const.}\ l/\cos\theta \cdot \exp(-l/\Lambda\cos\theta)(E_0^{-2.8} - 2.2E_0^{-3.5})(1 + (B/\cos\theta)/E_0)^{-1} \tag{3}$$

Then the differential energy-spectrum of μ-mesons, $f(E, l, \theta)$, can be written down, making some alterations for the integrand and the upper limit of the integral[6] appeared in Eq. (22), I, as follows:

$$f(E, l, \theta)dE = dE \int_E^{(\mu_\pi/\mu_\mu)^2 E} \frac{1}{1 - (\mu_\mu/\mu_\pi)^2} \cdot \frac{dE_0}{E_0} \int_0^l \frac{B/\cos\theta}{E_0 l'} \cdot g(E_0, l', \theta) dl'$$

$$= \text{const.}\ dE \cdot \Lambda \cdot (1 - \exp(-l/\Lambda\cos\theta)) \frac{1}{1 - (\mu_\mu/\mu_\pi)^2} \int_E^{(\mu_\pi/\mu_\mu)^2 E} \frac{(E_0^{-3.8} - 2.2E_0^{-4.5})}{1 + E_0/(B/\cos\theta)} dE_0. \tag{4}$$

As this equation is not correct for the lower energy region because of the neglect of all sorts of energy losses of both π- and μ-mesons, we fix the value of const. in this equation using the data of Rossi[7] for $f(E, 10, 0^\circ)$, the spectrum at sea-level for vertical incidence, at the energy of $E = 3 \times 10^{10}$ eV. Accordingly the integral energy spectrum of μ-mesons on the sea-level, $F(E, 10, \theta)$, becomes

$$F(E, 10, \theta) = \int_E^\infty f(E', 10, \theta) dE'. \tag{5}$$

We integrated (5) numerically for the values of $\theta = 0^\circ$, 15°, 30°, 45°, 60°, 75°, and 90° taking μ_μ, the mass of the μ-meson, to be 217 m.

On the other hand the minimum energy E required for μ-mesons coming from the direction θ to penetrate to the depth $x\,\mathrm{gcm}^{-2}$ from the surface of the earth is given, according to Eq. (12), I, by

$$E(x,\theta)=\frac{a}{p+r}[\exp\{(p+r)\cdot x/\cos\theta\}-1], \tag{6}$$

which takes account of the energy losses of μ-mesons due to ionization, pair creation, and radiation expressed by the following parameters:

$$a=2.5\times10^{6}\ \text{eV per g cm}^{-2},$$

$$p=1.6\times10^{-6}\quad \text{per g cm}^{-2},$$

$$r=1.0\times10^{-6}\quad \text{per g cm}^{-2}.$$

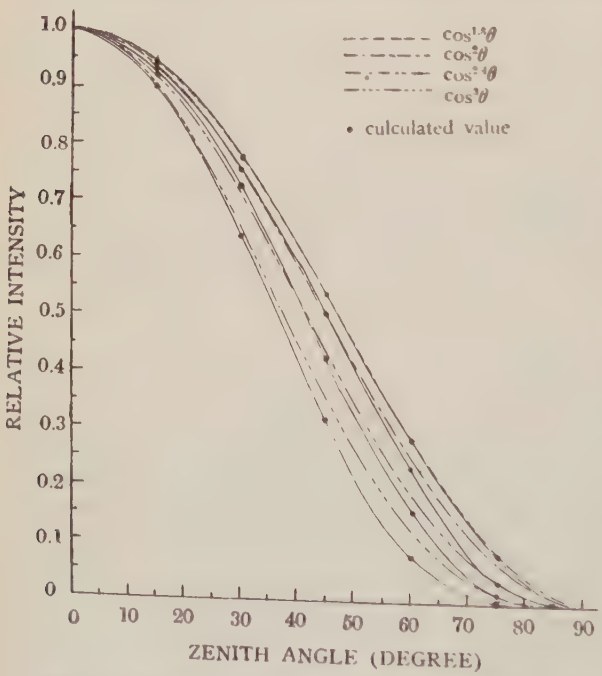
For given x and θ we have the directional intensity, $I(x,\theta)$, by substituting the value of E determined by (6) into (5):

$$I(x,\theta)=F(E(x,\theta),10,\theta). \tag{7}$$

When $x/\cos\theta$ is sufficiently small Eq. (6) may be approximated by

$$E(x,\theta)=a\cdot x/\cos\theta. \tag{6'}$$

As the energy is fairly large for $x>200$m w.e. (water equivalent) the second term of the spectrum $2.2E_0^{-4.5}$ has no contribution. This results in the $\cos^{1.8}\theta$ law reflecting the energy spectrum of π-mesons for the angular distribution irrespective of whether the depth under consideration is above or below the bend. This is because the effect of the transformation (1), appearing only in the term $\exp(-l/\Lambda\cos\theta)$ of (4) which being far small as compared with unity, is negligible for every direction on one hand, and the transformation (1'), appearing in the denominator of Eq. (4), just effects the cancellation of the factor $\cos\theta$ together with Eq. (6') on the other. Therefore, if any deviation from the $\cos^{1.8}\theta$ law may take place, it must be attributed to the fact that, at high energies, obliquely incident particles suffer more energy losses than those of vertical due to radiation and pair creation, and consequently, we may expect a sharper angular distribution



Solid curves represent downwards the calculated results for $x=200$, 730 1700, 3000 m w.e., respectively.

at a greater depth. The results of calculations illustrated in the figure show the correctness of our expectation. For $x=200$ m w.e. the angular distribution is exactly $\cos^{1.8}\theta$ reflecting the energy spectrum of the primary rays; for $x=730$ m w.e. it may be approximated as $\cos^{2}\theta$ and is a little steeper than the experimental result $\cos^{1.7}\theta$ of Barnóthy and Forro[4]; for $x=1700$ m w.e. it is about $\cos^{2.4}\theta$ and this seems to be in fairly good agreement with the Greisen's experimental result of about $\cos^{3}\theta$ at the same depth[2] when we take the experimental error into account. Although we have so far only one available experimental result of Greisen at 1700 m w.e. to testify crucially the deviation from the $\cos^{1.8}\theta$ law, it may serve, if this experiment is really true, as a strongest evidence for the validity of our equation (6) for the energy loss.

Thus it may be concluded that those rays which penetrate far underground are μ-mesons and that they interact with matter according to the ordinary quantum electrodynamics. Further the Greisen's opinion[8] that the phenomenon under consideration can be well explained by the neutrino hypothesis of Barnothy and Forró seems highly improbable.

§ 3. The temperature effect

The temperature effect of the hard component of the cosmic-ray with regard to the atmospheric temperature at the earth's surface has negative sign and is -0.15% per °C, whereas according to the Forró's experiment which was performed at the depth of 1000m w.e. the same effect with regard to the outer-air temperature is found to be $+0.74\%$ per °C. The positive sign of the coefficient under the ground might be regarded as a suppport for π-μ-decay because the higher the temperature, the more π-mesons decay into μ-mesons before reaching the sea-level due to the upward shift of the height at which π-meson production takes place, and consequently the more rays are produced which are penetrable into the ground. However, according to the simple calculations of Forró[3] and one of the authors[9] the underground temperature effect can not exceed the value $+0.16\%$ per °C. Although this is right as far as the qualitative argument is concerned, there seems to be a considerable numerical discrepancy between the theory and the experiment. We will examine here whether the result which agrees quantitatively well with the experiment can be derived from the idea of I.

Change of the temperature affects the cosmic-ray intensity through the factor B. When the temperature rises the air density ρ decreases, and accordingly increases B. The influence of increasing B on the cosmic-ray intensity can easily be seen from Eq. (4). The temperature coefficient has, of course, positive sign, since we are neglecting the decay of μ-mesons.

Hereafter we consider only the vertical intensity. At a point far deeper than the bend inequality $B < E_0$ holds and therefore, the denominator of Eq. (4) can be expanded as

$$\frac{B}{E_0}\left(1-\frac{B}{E_0}+\left(\frac{B}{E_0}\right)^2-\cdots\cdots\right).$$

Thus we see that the intensity is proportional to B at the most. The intensity-temperature relation may be written down as

$$\Delta I/I=\Delta B/B=-\Delta\rho/\rho=\Delta T/T. \tag{8}$$

Accordingly, if the temperature at the production layer of π-mesons is put for T, the temperature coefficient becomes

$$\frac{\Delta I}{I}\Big/\Delta T=1/T\approx 1/250=0.4\%\ \text{per}\ ^\circ\text{C}. \tag{9}$$

Generally, when the above-mentioned approximation breaks down, we can never have the temperature coefficient larger than this value. Is there any reason to explain the existing small gap between the theory and the experiment?

First of all the temperature used by Forró for the correlation was that of the earth's surface. On the other hand the temperature used in (8) is that at the height in the atmosphere where almost all the π-mesons decay into μ-mesons. Next, the temperature used in the approximation of I in which we neglected the absorption of π-mesons in the atmosphere meant the average taken for the region lower than the height of π-meson production. But, as regression coefficients among these three kinds of temperatures are found practically unity, the result does not depend upon whichever of these temperatures is taken for the correlation. Consequently, if the temperature coefficient obtained by Forró is really significant the cause of the gap can not be hit upon for the time being.

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Note added in proof: Our statement, that the angular distribution at great depth can be explained by the electromagnetic interaction of μ-mesons accounting for the bremsstrahlung and pair creation process, is not correct. The recent experiment has established much steeper distribution, $n=3.1\pm0.1$ for small θ, wheras our theory predicts $n\sim 2.4$. This discrepancy will be explained by the nuclear interaction of μ-mesons with the cross-section about 10^{-29}cm^2 per nucleon. We should like to express our gratitude to Professor Greisen for his helpful discussion.

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The Covariant Theory of Radiation Damping

(*I*) *General Formalism*

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§ 1. Introduction

The covariant form of the *S*-matrix has been derived by Dyson[1] using the Tomonaga equation. The elements of this matrix can be calculated, by a consistent use of perturbation theory, to any desired order in the fine structure constant. Feynman[2] and Dyson[1] gave detailed rules for carrying out such calculations, and it was shown that the divergencies arising from higher order radiative corrections could be removed from the *S*-matrix by a consistent use of the ideas of mass and charge renormalization. This formalism was adopted by Karplus and Kroll[3] in order to calculate the fourth order radiative correction to the magnetic moment of electron, and it was verified that the application of perturbation theory is really consistent in quantum electrodynamics.

The success of this renormalization program has given impetus to a re-examination of the difficulties in the meson theory. There are three questions to be asked in this connection: (1) Can the renormalization program of electrodynamics be successfully applied also to the meson theory? (2) In view of the largeness of the coupling constant describing the interaction of meson and nuclear fields, can one obtain valid solutions to problems concerning with these interactions? (3) Will the cross section for meson scattering and creation remain finite with increasing momentum, on account of the reactive and damping influence?

If we restrict ourselves to the (pseudo) scalar meson with (pseudo) scalar coupling, we can always obtain finite results, to any desired order, by the renormalization program. It has been pointed out, however, by many authors[4] that the renormalization of mass and mesonic charge is not sufficient to remove all divergences in any theory in which a gradient coupling is used. The second question above poses formidable analytical difficulties, since we have at present no other covariant method than the perturbation one in solving the Schrödinger equation. Case's[5] treatment of the anomalous magnetic moments of the nucleons, and the results obtained by Watson and Lepore[6] concerning the radiative corrections to nuclear forces, suggest the necessity for research about this second question.

Also the third question cannot be solved, so long as we apply the perturba-

tion method to the S-matrix. In this connection, we remember the Heitler and Wilson's theory of radiation damping[7]. As far as the meson problem is concerned, this theory gives reasonable results except that the scattering cross section becomes too small for high energies. But because in this theory no account is taken of virtual processes, not only the infra-red catastrophe remains unsolved[8], but also the anomalous magnetic moment of electron cannot be explained. It was shown by Pauli[9] that the Heitler's formalism could be generalized so as to include all the radiative corrections. If we could introduce the renormalization program into this theory, we shall get a reasonable formalism for treating all the scattering, emission and absorption processes[10].

The present paper deals with the general theory along this line. As the covariant formalism of Tomonaga and Schwinger has been used, the relativistic invariance of our theory is self-evident. If we apply the perturbation method directly to the S-matrix, the most important property, i.e., the unitary character of this matrix will be destroyed. On the contrary, this unitarity is strictly maintained in our theory in any approximation. This point must be stressed especially in the meson theory, because in this case the coupling constant may not be so small as in the electrodynamics. Many applications and discussions will be published in the subsequent papers in this journal.

§ 2. Formalism

We shall start from the Tomonaga equation

$$i\frac{\delta}{\delta\sigma(x)}\Psi[\sigma]=H(x)\Psi[\sigma], \tag{2.1}$$

where $H(x)$ is an invariant Hamiltonian density usually dependent also on the surface σ. (We use natural units throughout this work; $\hbar=c=1$.) The eq. (2.1) can be solved by the unitary matrix $U[\sigma]$ in the form

$$\Psi[\sigma]=U[\sigma]\Psi[-\infty]. \tag{2.2}$$

$U[\infty]$ is the so-called S-matrix[11]. If the perturbation theory can be applied, S is given by

$$S=\sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int_{-\infty}^{\infty}dx_1\ldots\ldots dx_n\ P(H(x_1)\ldots\ldots H(x_n)), \tag{2.3}$$

where the symbol P is the chronological operator defined by Dyson[1]. As was shown by Tomonaga and others, the self-energy matrix must be subtracted from the Hamiltonian density in order to carry out the mass renormalization consistently.

If we put

$$S=1-iR, \tag{2.4}$$

the elements of the matrix R represent the amplitude of the scattered wave. Since the S-matrix is not unitary if we content ourselves with the proper approximation in the power series (2.3), we shall perform the Cayley transformation of S:

$$S=\frac{1-\frac{i}{2}K}{1+\frac{i}{2}K}. \tag{2.5}$$

S is always kept unitary, even if we replace K by its approximate hermitian matrix. From the eqs. (2.4) and (2.5) we obtain the Heitler and Pauli's equation of radiation damping

$$R=K-\frac{i}{2}KR, \tag{2.6}$$

which is the fundamental equation we are going to treat.

The integral equation defining K was established by Schwinger[12], assuming the first order real process does not occur:

$$\begin{aligned} K&=\int_{-\infty}^{\infty}dxH(x)V[\sigma], \\ V[\sigma]&+\frac{i}{2}\int_{-\infty}^{\infty}dx'\ \varepsilon[\sigma,\sigma']\ H(x')\ V[\sigma']=1. \end{aligned} \tag{2.7}$$

If the successive approximation for $V[\sigma]$ is convergent, K is given by

$$K=\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}(-i)^{n-1}\ \int_{-\infty}^{\infty}dx_1\cdots\cdots dx_n\ H(x_1)\cdots\cdots H(x_n)\varepsilon[\sigma_1,\sigma_2]\cdots\cdots\varepsilon[\sigma_{n-1},\sigma_n]. \tag{2.8}$$

Denoting the n-th order term of (2.3) and (2.8) by $S^{(n)}$ and $K^{(n)}$, respectively, and the anti-hermitian part of $S^{(n)}$ by $S_i^{(n)}$, we get the relations

$$K=iS_i^{(n)} \qquad (\text{for } n\leqq 5). \tag{2.9}$$

Since the total energy and momentum are conserved, the matrix elements of S, R and K must have the form

$$(p^1,\cdots\cdots,p^n|\ A\ |\ p_0^1,\cdots\cdots,\ p_0^{n_0})=\delta^4(P-P_0)\,(p^1,\cdots\cdots,p^n|\ \bar{A}\ |\ p_0^1,\cdots\cdots,p_0^{n_0}), \tag{2.10}$$

where

$$P_0=\sum_{i=1}^{n_0}p_0^i \quad \text{and} \quad P=\sum_{i=1}^{n}p^i \tag{2.11}$$

are the energy-momentum four vectors of the initial and final state, respectively, and $\bar{A}$ is a submatrix of A belonging to a fixed value of $P=P_0$. Denoting simply $(p|\ \bar{A}\ |\ p_0)$ for the matrix elements of $\bar{A}$, the fundamental equation (2.5) is written down in the more convenient form

$$(p|\bar{R}|p_0)=(p|\bar{K}|p_0)-\frac{i}{2}\int(p|\bar{K}|q)(q|\bar{R}|p_0)\ II\ \frac{d^4q}{(2\pi)^3}\ \delta[q^2+M^2]\delta^4(Q-P_0), \quad \text{(I)}$$

where the integration covers all the final states (including the number of particles) that can be reached from the initial state.

We shall often consider a collision between particle 1 (p_0^1) and 2 (p_0^2). After the collision, these particles will take the energy-momentum p^1 and p^2 respectively, creating a number of new particles ($k^1,\ldots\ldots,k^m$). The invariant differential cross section for this process was defined by Møller[13].

$$d\sigma=\frac{1}{\varphi(p_0^1,\ p_0^2)}\ |(p^1,\ldots\ldots,k^m\mid \bar{R}\mid p^1_0,\ p^2_0)|^2\,\rho(p^1,\ldots\ldots,k^m), \tag{II}$$

where

$$\varphi=\sqrt{-\frac{1}{2}(p^1{}_0\times p^2_0)\cdot(p_0^1\times p_0^2)} \tag{2.12}$$

is the flux of incident particles, and

$$\rho=\frac{d^4p^1}{(2\pi)^3}\cdots\cdots\frac{d^4k^m}{(2\pi)^3}\ \delta[(p^1)^2+(M^1)^2]\cdots\cdots\delta[(k^m)^2+(\mu^m)^2](2\pi)^{-4}\delta^4(P-P_0) \tag{2.13}$$

is the so-called statistical factor. $a\times b$ is a tensor whose components are given by $(a_\mu b_\nu-a_\nu b_\mu)$.

§3. The Feynman and Dyson method in the meson theory

To describe the meson field we choose the simplest possibility which may be expected to yield interesting results—namely, the pseudoscalar theory. In the case of interacting meson and nuclear fields, the invariant Hamiltonian density H is given by

$$H=if_k\ \bar{\psi}(x)\tau_k\gamma_5\ \psi(x)\ \phi_k(x)+i\frac{g_k}{\mu_k}\ \bar{\psi}(x)\ \tau_k\gamma_5\gamma_\mu\frac{\partial\phi_k(x)}{\partial x_\mu}$$
$$-\frac{1}{2}\frac{g_k^2}{\mu_k^2}\ (\bar{\psi}(x)\ \tau_k\gamma_5\psi(x)\,u_\mu)^2-\delta M\bar{\psi}(x)\psi(x)-\delta\mu_k^2\ \phi_k(x)\ \phi_k(x), \tag{3.1}$$

where the dummy indices k and μ mean the summation from 1 to 4. τ_1, τ_2, τ_3 are the conventional isotopic spin matrices and τ_4 is merely the unit matrix in the isotopic spin space.

The vacuum expectation values are

$$<P(\phi_i(x),\ \phi_k(x'))>_0=-i\delta_{ik}\ \Delta_{Fk}\ (x-x'), \tag{3.2}$$
$$<P(\bar{\psi}_\alpha(x),\ \psi_\beta(x'))>_0=-i\ S_{F\beta\alpha}\ (x-x') \tag{3.2}$$

where the index α refers to both spinor and isotopic spin components, and $\Delta_{Fk}(x)$ and $S_F(x)$ are $\frac{i}{2}$ times those given by Feynman, except that S_F is a unit matrix with respect to isotopic spin indices; namely,

$$\Delta_{Fk}(x)=\frac{1}{(2\pi)^4}\int d^4k\ e^{-ikx}\Delta_{Fk}(k), \qquad \Delta_{Fk}(k)=\frac{1}{k^2+\mu_k^2-i\varepsilon}, \tag{3.3}$$

$$S_F(x)=\frac{1}{(2\pi)^4}\int d^4p e^{-ipx}\ S_F(p), \qquad S_F(p)=\frac{ip-M}{p^2+M^2-i\varepsilon}, \tag{3.4}$$

where ε is a positive, small constant which will not be written explicitly in the following. As was demonstrated by Matthews[14], the third term in the Hamiltonian may be omitted if we put effectively

$$<P\left(\frac{\partial\phi_i(x)}{\partial x_\mu}, \frac{\partial\phi_k(x')}{\partial x_\nu'}\right)>_0=\frac{\partial^2}{\partial x_\mu\partial x_\nu'}<P(\phi_i(x), \phi_k(x'))>_0. \tag{3.5}$$

According to Feynman and Dyson, the matrix element of S in question is given by a sum of term each of which may be described uniquely by a graphically represented transition scheme. For any given transition diagram G, the corresponding matrix element in the momentum space is obtained by the following prescriptions.

(i) The coefficient is

$$(-1)^{n-m-l}\ (-i)^p\ (2\pi)^{4(n-p)}\ \delta^4(P-P_0), \tag{3.5}$$

where n, m, l, and p indicate the number of vertices, of open polygonal arcs o nucleons, of loops, and of internal nucleon lines, respectively.

(ii) Corresponding to each external line of G, we put a constant spinor or pseudoscalar

$$\psi(p),\ \bar{\psi}(p),\ \ \phi_i(k) \text{ or } \phi_i(k), \tag{3.6}$$

whose normalizations are given by

$$\bar{\psi}(p)\gamma_4\psi(p)=E(p), \tag{3.7}$$

$$\phi_i^*(k)\phi_i(k)=\frac{1}{2} \quad \text{(for each } i\text{)}, \tag{3.8}$$

$E(p)$ being the energy of nucleon.

(iii) Corresponding to each internal line of G, we put a factor

$$\delta_{ik}\Delta_{Fk}(k'),\ \ S_F(p'). \tag{3.9}$$

(iv) Corresponding to each vertex of G, we put a factor

$$F_i\equiv f_i\tau_i\gamma_5 \quad \text{(for pseudoscalar coupling)}, \tag{3.10}$$

$$G_i\equiv\frac{g_i}{\mu_i}\tau_i\gamma_5\ (\pm i\gamma k') \quad \text{(for pseudovector coupling)}, \tag{3.11}$$

where the + or − sign is used according as the meson annihilates or creates at the vertex, k' being its energy-momentum four vector. When the self-energy term of (3.1) is operating, we have to put

$$i\delta M \quad \text{or} \quad i\delta\mu^2 \tag{3.12}$$

The virtual energy and momentum should of course be conserved at each vertex. After the above calculations, we must perform the integrations over some energy-momentum variables which may take arbitrary values. If these integrals are divergent, however, the result may depend on which energy-momentums we shall take as the integral variables. When we want to get the matrix element of $K^{(n)}$, it is necessary to calculate the anti-hermitian part of $S^{(n)}$. It will generally be shown that this anti-hermitian part is obtained by taking the principal values of the integrals over energy-momentum variables at the poles[9].

§4. The renormalization of mass and mesonic charge

This section is concerned with the evaluation of modified functions to second order of S_F, D_F and F_i in the case of pure pseudoscalar coupling. The calculations including both the pseudoscalar and pseudovector couplings are so much complicated that they will be published in the subsequent papers. After we have carried out mass and mesonic charge modification in these modified functions, all the Feynman diagrams, which contribute to the matrix element in question containing second order corrections, will have direct physical significance.

Since the evaluation is almost the same as that performed by Karplus and Kroll[3], we shall mention here only the results.

(a) The function $S'_F(p)$, which describes the property of a virtual nucleon as modified by its interaction with the meson field, must be calculated to second order. The leading term, of course, is $S_F(p)$. The second order correction $S_F^{(2)}(p)$ to this function arises both from the ability of the virtual nucleon to create and absorb a meson, and from the nucleon self-energy term (3.1), as described by the Feynman diagram Fig. 1.

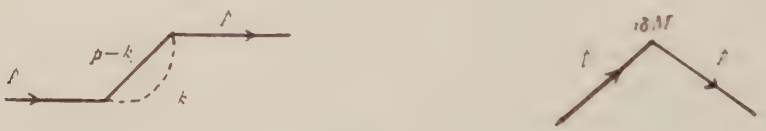

Fig. 1.

After the mass and mesonic charge have been renormalized, $S_F^{(2)}(p)$ becomes $\bar{S}_F(p)$ which has direct physical significance. The result of calculation shows

$$\bar{S}_F(p)=\frac{1}{4\pi}\sum_i\left(\frac{f_i^2}{4\pi}\right)\int_0^1 du\; u(1-u)\int_0^1 dv\;\frac{Mu+(i\gamma p-M)(1-u)\left\{1-\dfrac{2M^2u^2v}{\Lambda_i}\right\}}{\Lambda_i+(p^2+M^2)u(1-u)v}, \tag{4.1}$$

where Λ_i is given by

$$\Lambda_i=M^2u^2+\mu_i^2(1-u). \tag{4.2}$$

(b) Quite similarly, the second order correction $\Delta_{Fi}^{(2)}(R)$ to $\Delta_{Fi}(R)$ is described by the Feynman diagram Fig. 2. The physically significant part $\Delta_{Fi}(k)$ of $\Delta_{Fi}^{(2)}(k)$

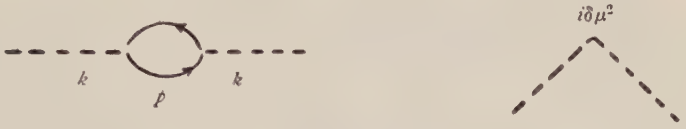

Fig. 2.

is given by

$$\Delta_{Fi}(k)=\frac{4}{\pi}\sum_{k}\left(\frac{f_k^2}{4\pi}\right)\int_0^1 du\ u^2(1-u)^2\int_0^1 dv\times$$

$$\times\left\{\frac{M^2}{\Lambda_i'[\Lambda_i'+(k^2+\mu_i^2)u(1-u)v]}+\int_0^1 dw\,\frac{1}{\Lambda_i'+(k^2+\mu_i^2)u(1-u)vw}\right\},\quad(4.3)$$

where Λ_i' is defined by

$$\Lambda_i'=M^2-\mu_i^2\ u(1-u).\qquad(4.4)$$

(c) The second order correction $L_i^{(2)}(t^1, t^2)$ to the vertex matrix F_i is described by the Feynman diagram Fig. 3.

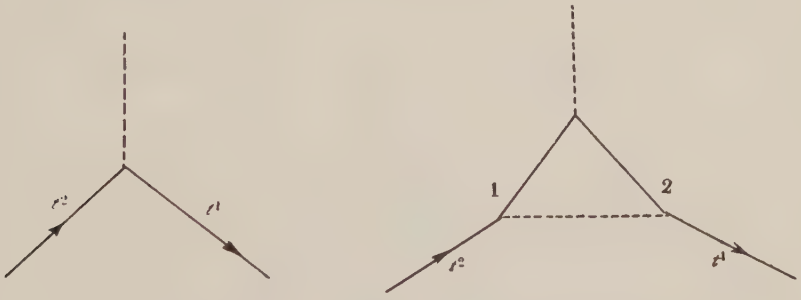

Fig. 3.

$L_i^{(2)}(t^1, t^2)$ can be written in the form

$$L_i^{(2)}(t^1, t^2)=A_i\ F_i+\bar{L}_i(t^1, t^2),\qquad(4.5)$$

where A_i is the divergent integral which may be interpreted as the mesonic charge renormalization term. This separation is not unique however, because we have no general principle to do so. Watson and Lepore[6] carried out this separation in such a way that $\bar{L}_i(t^1, t^2)$ becomes zero if the nucleons t^1, t^2 and the meson k should be free simultaneously—which, in fact, is impossible, for $k=\pm(t^1-t^2)$ on account of the energy-momentum conservation at the vertices. In this case, $\bar{L}_i(t^1, t^2)$ is given by

$$L_i(t^1, t^2)=-\frac{1}{4\pi}\sum_{k}\left(\frac{f_k^2}{4\pi}\right)T_{ik}\tau_i\int_0^1 du u\int_0^1 dv\ \times$$

$$\times\left\{\frac{K_i}{\Lambda''_{ik}}+[M^2-\mu_i^2\ v(1-v)]u^2\frac{\Lambda''_{ik}-\Lambda''_{iko}}{\Lambda''_{ik}\ \Lambda''_{iko}}\gamma_5\right. \tag{4.6}$$

$$\left.-2\int_0^1 dw\frac{\Lambda''_{ik}-\Lambda''_{iko}}{\Lambda''_{iko}+(\Lambda''_{ik}-\Lambda''_{iko})w}\gamma_5\right\}$$

where

$$\Lambda''_{ik}=M^2\ u^2+\mu_k^2(1-u)-\mu_i^2\ u^2\ v(1-v)$$
$$+(k^2+\mu_i^2)u^2\ v(1-v)+[(t^1)^2+M^2)v+((t^2)^2+M^2)(1-v)], \tag{4.7}$$

$$\Lambda''_{iko}=M^2\ u^2+\mu_k^2(1-u)-\mu_i^2\ u^2\ v(1-v), \tag{4.8}$$

$$K_i=(i\gamma\ t^1+M)\gamma_5(i\gamma\ t^2+M)(1-u)$$
$$+(i\gamma\ t^1+M)\gamma_5 Mu+\gamma_5(i\gamma\ t^2+M)Mu$$
$$-(k^2+\mu_i^2)u^2v(1-v)\gamma_5$$
$$-[((t^1)^2+M^2)v+((t^2)^2+M^2)(1-v)]u(1-u)\gamma_5, \tag{4.9}$$

$$T_{ik}=1 \qquad \text{for } i=4 \text{ or } k=4$$
$$=2\delta_{ik}-1 \quad \text{for } i\neq 4 \text{ and } k\neq 4. \tag{4.10}$$

We notice here that in the charged meson theory $L_i^{(2)}(t^1, t^2)$ is identically zero.

§5. An application to the meson scattering

As well known, the cross section for the meson scattering increases rapidly with the momentum of the incident meson. Many authors have shown that this difficulty could be overcomed by taking into account the reaction and damping influence of the meson field. It must be pointed out, however, that the calculations so far were either non relativistic neglecting entirely the recoil of the nucleon, or introduced the approximation which might be considered not always appropriate[15]. We are, therefore, going to treat the influence of damping on the meson scattering, as an application of the fundamental equation (I). The radiative corrections and the multiple processes arising from $\bar{K}^{(3)}$, $\bar{K}^{(4)}$, etc., will be neglected here, which are to be published soon in the following papers. The charged meson theory was chosen, because in this case the above integral equations can be solved exactly.

Denoting the initial and final energy-momentum of the nucleon and meson by p_0, k_0 and p, k, respectively, the matrix element of $\bar{K}$ for the meson scattering (not containing the multiple process) has the general form

$$(p,\ k|\bar{K}|p_0,\ k_0)=(2\pi)^4\ \phi_k^*(k)\phi_i(k_0)\{\bar{\psi}(p)\tau_k\tau_i A\ (p,\ k;\ p_0,\ k_0)\psi(p_0)$$
$$+\bar{\psi}(p)\tau_i\tau_k B(p,\ k;\ p_0,\ k_0)\psi(p_0)\}. \tag{5.1}$$

If we put

$$(p, k|\bar{R}|p_0, k_0) = (2\pi)^4 \phi_k^{\dagger}(k)\phi_i(k_0)\{\bar{\psi}(p)\tau_k\tau_\tau X(p, k; p_0, k_0)\psi(p_0) + \bar{\psi}(p)\,\tau_i\,\tau_k Y(p, k; p_0, k_0)\,\psi(p_0)\} \tag{5.2}$$

and introduce them in the equation (I), we shall get a pair of integral equations

$$X(p, k; p_0, k_0) = A(p, k; p_0, k_0) - \frac{il}{(4\pi)^2 W}\int d\Omega' A(p, k; p', k') \Lambda(p') X(p', k'; p_0, k_0), \tag{5.3}$$

$$Y(p, k; p_0, k_0) = B(p, k; p_0, k_0) - \frac{il}{(4\pi)^2 W}\int d\Omega' B(p, k; p', k') \Lambda(p') Y(p', k'; p_0, k_0) \tag{5.4}$$

for the unknown functions X and Y, where we have referred to the center of mass system, l is the absolute value of the momentum of the meson or nucleon, W is the total energy of both particles, and $\Lambda(p')$ is an annihilation operator defined by

$$\Lambda(p') = -i\gamma p' + M. \tag{5.5}$$

It is to be noted here the eqs. (5.3) and (5.4) have nothing to do with each other, which is not the case in the symmetrical theory where the neutral meson may appear in the final states.

If we retain only the second order term $K^{(2)}$ of K, A and B are given by

$$A^{(2)} = \frac{a_1 - a_2\gamma_4}{(p_0 + k_0)^2 + M^2}, \tag{5.6}$$

$$B^{(2)} = \frac{b_1 + b_2 i\gamma(p_0 - k)}{(p_0 - k)^2 + M^2}, \tag{5.7}$$

where

$$a_1 = f^2 M + \frac{2fg}{\mu}(M^2 + W^2) + \frac{g^2}{\mu^2}(M^3 + 3MW^2), \tag{5.8}$$

$$a_2 = f^2 W + \frac{4fg}{\mu} MW + \frac{g^2}{\mu^2}(W^3 + M^2 W), \tag{5.9}$$

$$b_1 = f^2 M + \frac{2fg}{\mu}[M^2 - (p_0 - k)^2] + \frac{g^2}{\mu^2}[M^3 - M(p_0 - k)^2], \tag{5.10}$$

$$b_2 = f^2 + \frac{4fg}{\mu} M + \frac{g^2}{\mu^2}[M^2 - (p_0 - k)^2]. \tag{5.11}$$

The eq. (5.3) describes the scattering of Y^+ by N (neutron) or Y^- by P (proton), while the eq. (5.4) does that of Y^+ by P or Y^- by N. Since the latter integral equation depends on the angle of scattering, it cannot be solved exactly. Solving the equation with respect to X in the first process, we get the result

$$X=\frac{A}{1+i\chi}, \tag{5.12}$$

where

$$\chi=\frac{2lA}{(4\pi)^2W}(E\gamma_4+M) \tag{5.13}$$

represents the influence of damping, E being the energy of nucleon. Note that in the center of mass system the energy of nucleon or meson (ϵ) does not change after the scattering.

The differential cross section (II) becomes in this case

$$d\sigma=\frac{d\Omega}{W^2}\frac{1}{(2\pi)^2}\frac{1}{8}T_r(\Lambda(p)X\Lambda(p_0)\gamma_4X^\dagger\gamma_4), \tag{5.14}$$

where we have taken the average about the initial spin and the summation over the final spin of the nucleon. Performing the integration over the angle of scattering and putting

$$F=\frac{1}{\sqrt{4\pi}}\left(f+\frac{2M}{\mu}g\right), \qquad r=\frac{\frac{2M}{\mu}g}{\sqrt{4\pi}F}, \tag{5.15}$$

we obtain the total cross section

$$\sigma=\frac{8\pi}{M^2}F^4\frac{\zeta(x)}{\pi(x)}, \tag{5.16}$$

where

$$\zeta(x)=(1+2\rho T(x))\xi(x)+F^4\rho^2x^6\}-1+2r+r^2(-1+\rho T(x))-r^3\rho T(x)$$
$$-(r^4\rho^2/16)(\rho+2T^2(x))^2\}^2, \tag{5.17}$$

$$\pi(x)=\{(\rho+2T(x))(1+2\rho T(x))+F^4\rho^3x^4[-1+2r+r^2(-1+\rho T(x))$$
$$-r^3\rho T(x)-(r^4\rho^2/16)(\rho+2T^2(x))^2]\}^2$$
$$+F^4\rho^2x^2(1+2\rho T(x))^2\}-2T(x)+r(\rho+2T(x))(4-r\rho^2T(x))\}^2, \tag{5.18}$$

$$\xi(x)=(1+x^2)+T^2(x)-r(\rho+2T(x))\{(1+x^2)^{\frac{1}{2}}[2(1+\rho^2x^2)^{\frac{1}{2}}$$
$$-(r\rho/2)(1+x^2)^{\frac{1}{2}}-r(1+\rho^2x^2)^{\frac{1}{2}}]+T(x)(2-r-(r\rho/2)T(x)\}$$
$$+(r^2/4)(\rho+2T(x))^2\}4(2+\rho^2x^2)+4r\rho T(x)-2r\rho^2x^2$$
$$+(r\rho/2)^2(1+x^2+T^2(x))-2r(4-r)(1+\rho T(x))\}, \tag{5.19}$$

$$\rho=\frac{\mu}{M}\fallingdotseq 0.15, \qquad x=\frac{l}{\mu}, \tag{5.20}$$

and $\mu T(x)$ is the energy (ϵ) of the incident meson in the laboratory system, namely

$$T(x)=\frac{\bar{\epsilon}}{\mu}=\frac{k^2+\epsilon E}{\mu M}=\rho x^2+(1+x^2)^{\frac{1}{2}}(1+\rho^2 x^2)^{\frac{1}{2}}. \tag{5.22}$$

The values of $T(x)$ for a number of x are given in the Table I.

x	1	2	3	4	5
$T(x)$	1.57	2.91	4.82	7.17	10.10

Table 1.

When we neglect the influence of damping, the cross section becomes

$$\sigma^{(2)}=\frac{8\pi}{M^2}F^4\frac{\xi(x)}{\eta(x)}, \tag{5.22}$$

where $\xi(x)$ is given by (5.19) and

$$\eta(x)=(1+2\rho T(x))(\rho+2T(x))^2. \tag{5.23}$$

Table II gives the cross section $\sigma^{(2)}$ for different values of x and r when $F=1$. For other values of F multiply only the factor F^4 to this $\sigma^{(2)}$. $r=0$ corresponds to the pure pseudoscalar coupling and $r=1$ to the pure pseudovector coupling. Due to the influence of the recoil of nucleon, the cross section does not increase even if x reaches as much as 5, or in the laboratory system, the energy of the incident meson amounts to 1.5 times the nucleon mass. This is contrary to our usual expectation.

x \ r	0	1/2	1	2
1	3.27×10^{-27}	0.00×10^{-27}	6.13×10^{-27}	80.0×10^{-27}
2	2.28	0.04	5.94	48.4
3	1.58	0.23	5.79	47.5
4	1.13	0.37	5.89	48.5
5	0.80	0.46	5.94	49.8

Table II. Values of $\sigma^{(2)}$. $F=1$.

Table III_1 and III_2 give the ratio $\sigma/\sigma^{(2)}$ for $r=1$ and $r=2$, respectively. $r=2$ corresponds to the case where f and g have the opposite signs. In the other two cases, the influence of damping is negligibly small even at $x=5$. As expected, the bigger the coupling constant and the momentum, the greater the

influence of damping.* It is to be noted that the cross sections $\sigma^{(2)}$ and σ do not approach to zero but to a finite value as l diminishes, as was first pointed out by Araki[15].

F^2 \ x	1	2	3	4	5
0.1	1.00	1.00	1.00	0.99	0.99
0.5	1.00	0.97	0.92	0.81	0.69
1.0	0.98	0.88	0.74	0.52	0.35
10.0	0.26	0.067	0.032	0.016	0.032

Table III_1. The ratio $\sigma/\sigma^{(2)}$ for $r=1$.

F^2 \ x	1	2	3	4	5
0.1	1.00	0.99	0.97	0.92	0.83
0.5	0.94	0.74	0.54	0.30	0.17
0.0	0.79	0.42	0.23	0.10	0.047
10.0	0.036	0.007	0.003	0.001	0.005

Table III_2. The ratio $\sigma/\sigma^{(2)}$ for $r=2$.

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* In the pure pseudovector coupling, $F^2=1$ corresponds to $\frac{g^2}{4\pi}=0.0056$. Araki took $\frac{g^2}{4\pi}=0.05$, which may be considered too large.

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On the Anomalous Magnetic Moment of Meson

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§ 1. Introduction

Using the Schwinger-Tomonaga formalism, corrections to the meson current due to interaction with the nucleon field have been calculated, and that from motives chiefly of the following considerations.

(1) The calculation of nuclear forces and the anomalous magnetic moment of nucleon, applying the Schwinger-Tomonaga formalism to meson theory, has been performed by many authors, the results obtained however being not in quantitative or even qualitative agreement with experiment. There are indeed some attempts to get rid of this difficulty, e.g. by modifying the model or by advancing the calculation to higher order approximations. But it may also be worth trying to attack the problem from a different aspect, namely to deal with *real* meson problems, instead of being concerned with *virtual* mesons, which has been the essential phase of the the theories in question.

(2) Recently, Kinoshita and Nambu[1] gave the corrections to the meson current due to electromagnetic field, and pointed out that vector or pseudo-vector meson should have, besides the anomalous magnetic moment, an intrinsic quadrupole moment. If however such a quantity had any real meaning, it would have been necessary to take account of the interaction with the nucleon field, which is stronger than that with the electromagnetic field.

§ 2. Method of calculation

The calculation proceeds according to the method used by Case[2] in deriving the magnetic moment of nucleon. Thus for the requisite effective Hamiltonian limited to the second order approximation, we have

$$H'_{eff}(x_0)=\frac{1}{2!}\left(\frac{-i}{\hbar c}\right)^2\int\int_{-\infty}^{+\infty}dx_1\ dx_2\ P\Big(H^{ext}(x_0),\ H_i(x_1),\ H_i(x_2)\Big), \tag{1}$$

where P denotes the chronological ordering opertor, and

$$H^{ext}=H_1^{ext}+H_2^{ext}, \tag{2}$$

$$H_1^{ext}=-ie\ \psi\tau_p\gamma_\mu\psi A_\mu, \tag{3}$$

$$H_2^{ext} = -j_\mu A_\mu, \quad (4)$$

$$j_\mu = \frac{ie}{\hbar c}\left(\frac{\partial\phi}{\partial x_\mu}\phi - \frac{\partial\phi}{\partial x_\mu}\phi\right), \text{ (scalar or pseudoscalar meson)} \quad (5)$$

$$= \frac{ie}{\hbar c}\left(\chi_{\mu\nu}\phi_\nu - \phi_\nu\chi_{\mu\nu}\right), \text{ (vector or pseudovector meson)} \quad (5')$$

$$\chi_{\mu\nu} = \frac{\partial\phi_\nu}{\partial x_\mu} - \frac{\partial\phi_\mu}{\partial x_\nu}.$$

Here e is the proton's charge, $\bar{\psi}$ and ψ are the spinor operators of the nucleon field, γ_μ the usual Dirac γ's, and τ_P is the isotopic spin operator for proton. A_μ denotes the four-vector potential of the external electromagnetic field, and ϕ, ϕ and ϕ_μ, $\bar{\phi}_\mu$ describe the charged scalar (pseudoscalar) and vector (pseudovector) meson field, respectively. H_i represents the interaction Hamiltonian between nucleon and meson fields, and, e.g. for the scalar coupling of scalar meson, may assume the form

$$H_i = f\bar{\psi}\,\tau_{PN}\,\psi\phi + \text{complex conjugate},$$

where f denotes the coupling constant. In the following sections, we shall use f for scalar and pseudoscalar couplings, g for vector and pseudovector couplings, and g' for tensor and pseudotensor couplings. τ_{PN} is the isotopic spin operator changing neutron into proton.

Now, our problem is to take the zero nucleon, one meson portion of the effective Hamiltonian. When limited to the second approximation, it can readily be seen that the meson current part of H^{ext}, i.e. H_2^{ext}, need not be considered,

Fig. 1

*) The term of order e^2 is neglected.

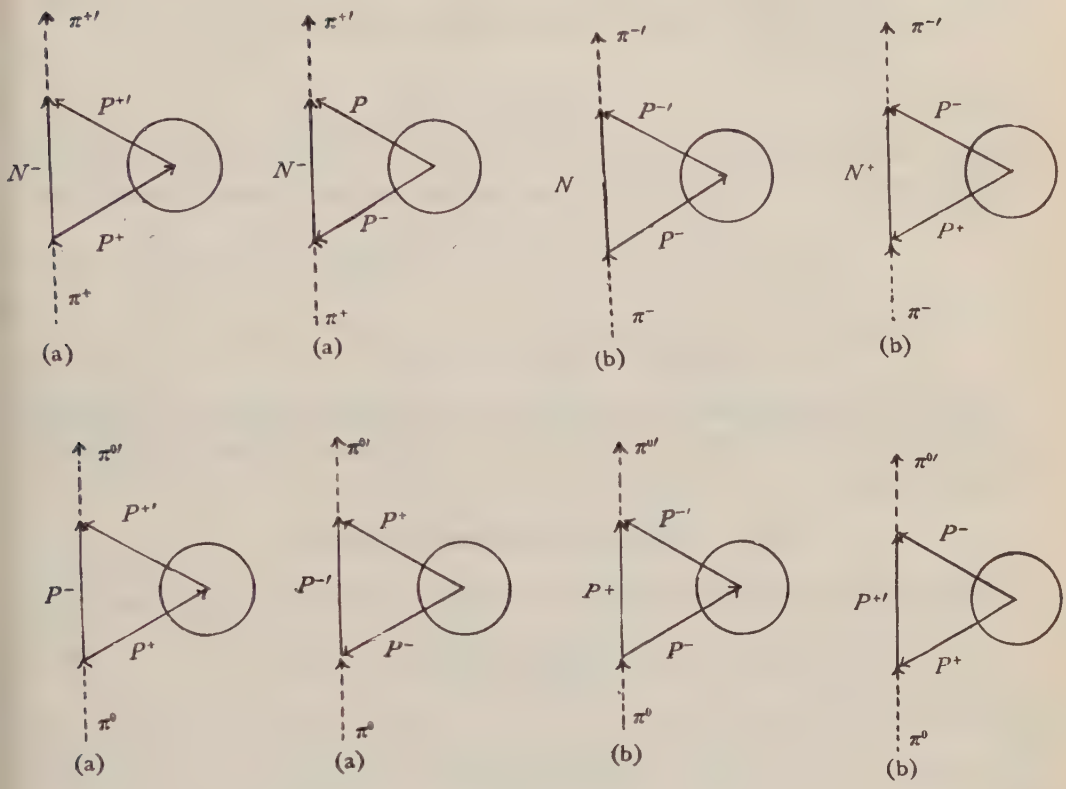

Fig. 2. Fig. 3. Fig. 4.

as this part can contribute only to higher approximations than the second. This situation makes the calculation simpler than that for nucleon. The Dyson diagram of the nucleon current part H_1^{ext}, up to the second approximation, is shown in Fig. 1, (a) and (b). They differ from each other only in the direction of nuclear lines. Detailed processes for positive (π^+), negative (π^-) and neutral (π^0) mesons are illustrated in Fig. 2, 3 and 4. In these figures P^+, N^+, P^- and N^- denote proton, neutron, antiproton and antineutron, respectively. Circles mean the external electromagnetic field. It appears from these figures that positive meson processes belong to the (a) diagram, negative ones to the (b) diagram and, finally, neutral mesons have processes which belong to both (a) and (b). But the (a) and (b) processes have the matrix element possessing equal magnitude and opposite signs, so that the neutral meson has, after all, no correction current. For this reason, we did not take the interaction Hamiltonian H_i for the neutral meson into account.

Thus, the portion of zero nucleon, one *positive* meson, in the case of scalar or pseudoscalar meson with scalar or pseudoscalar coupling, comes out to be

$$H=\pm\frac{ief}{8\hbar^2c^2}A_\mu\int_{+\infty}^{+\infty}\!\!\int dx_1\ dx_2\ \bar{\phi}(x_1)\ \phi(x_2)\ Tr\Big[S_F(x_0-x_1)\tau_P\gamma_\mu S_F(x_2-x_0)$$

$$\tau_{PN}(\gamma_5)\,S_F(x_1-x_2)\tau_{NP}(\gamma_5)\Big], \tag{6}$$

where S_F denotes Feynman's function for nucleon, whose Fourier transform is

$$S_F(x)=2\frac{i}{(2\pi)^4}\int d^4k\ \exp(ikx)\ \frac{ik\gamma+\varkappa_0}{k^2+\varkappa_0^2-i\varepsilon},^{*)} \tag{7}$$

$$\varkappa_0=\frac{Mc}{\hbar}, \tag{8}$$

in which M is the nucleon mass. In the double sign of (6), the upper sign corresponds to scalar meson, and the lower to pseudoscalar meson. (γ_5) is unity for scalar meson.

§ 3. Scalar and pseudoscalar mesons

As usual, we consider a typical Fourier component of field operators. Let

$$\phi(x)=\phi(p)\exp(ipx),\quad \bar{\phi}(x)=\bar{\phi}(p')\exp(-ip'x). \tag{9}$$

p_μ and p_μ' satisfy

$$p^2=p'^2=-\varkappa^2, \tag{10}$$

where

$$\varkappa=\frac{\mu c}{\hbar}, \tag{11}$$

μ being the π meson mass. Inserting (7) and (9) in the expression (6), noting that

$$Tr(\tau_P\ \tau_{PN}\ \tau_{NP})=1$$

$$Tr[(ik\gamma+\varkappa_0)\gamma_\mu(ik'\gamma+\varkappa_0)(\gamma_5)(ik''\gamma+\varkappa_0)(\gamma_5)]$$

$$=\mp ik_\lambda\ k_\sigma'\ k_\nu''\ Tr(\gamma_\lambda\gamma_\mu\gamma_\sigma\gamma_\nu)+4i\varkappa_0^2\ (k_\mu+k_\mu'\pm k_\mu''),$$

$$Tr(\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\sigma)=4(\delta_{\lambda\mu}\delta_{\sigma\nu}-\delta_{\sigma\mu}\delta_{\lambda\nu}+\delta_{\mu\nu}\delta_{\lambda\sigma}),$$

and integrating with respect to (x_1, x_2) and (k', k''), we finally obtain

$$H=-\frac{4ief^2}{\hbar^2c^2(2\pi)^4}A_\mu\int d^4k\frac{(k'k'')k_\mu+(kk')k_\mu''-(kk')k_\mu''\mp\varkappa_0^2(k_\mu+k_\mu'\pm k_\mu'')}{(k^2+\varkappa_0^2)(k'^2+\varkappa_0^2)(k''^2+\varkappa_0^2)}\times$$

$$\times\bar{\phi}(p')\phi(p)\exp(-i\Delta p\ x_0), \tag{12}$$

*) In place of scalar products $k_\mu x_\mu$, $k_\mu\gamma_\mu$ and k_μ^2, we adopted the conventions kx, $k\gamma$ and k^2.

with

$$k_\mu' = k_\mu + (\Delta p)_\mu, \quad k_\mu'' = k_\mu + p_\mu' \tag{13}$$

and

$$(\Delta p)_\mu = p_\mu' - p_\mu. \tag{14}$$

In the same manner as has been done by Feynman, (12) may be integrated over k-space, and be expressed as follows:

$$H = -\frac{ef^2}{8\pi^2\hbar^2c^2} A_\mu \int_0^1 dx \int_0^x dy$$

$$\left[\frac{\{\lambda(x-y)^2(x-y-1) \mp (2(y-x) \pm (1-x+y))\} y(1-x)\,(\Delta p/\varkappa_0)^2}{\Phi(x,y)\{\Phi(x,y) + (\Delta p/\varkappa_0)^2 y(1-x)\}} \right.$$

$$\left. - \frac{y(1-x+xy-x^2)\,(\Delta p/\varkappa_0)^2}{\Phi(x,y) + (\Delta p/\varkappa_0)^2 y(1-x)} + (3(y-x)-1) \ln \frac{\Phi(x,y) + (\Delta p/\varkappa_0)^2 y(1-x)}{\Phi(x,y)} \right] (p+p')_\mu \times$$

$$\times \bar{\phi}(p')\phi(p) \exp(-i\Delta p x_0), \tag{15}$$

where

$$\Phi(x,y) = 1 + \lambda(x-y)(x-y-1), \tag{16}$$

$$\lambda = \left(\frac{\varkappa}{\varkappa_0}\right)^2 = \left(\frac{\mu}{M}\right)^2. \tag{17}$$

Returning to the interpretation of ip_μ, $-ip_\mu'$ and $-i(\Delta p)_\mu$ as operating on the general wave functions, i.e. $ip_\mu \to \dfrac{\partial\phi(x)}{\partial x_\mu}$, $-ip_\mu' \to \dfrac{\partial\phi(x)}{\partial x_\mu}$, and $-i(\Delta p)_\mu \to \dfrac{\partial}{\partial x_\mu}$ $(\bar{\phi}(x)\phi(x))$, the expression (15) reduces to

$$H = -\frac{ef^2}{8\pi^2\hbar c} A_\mu \int_0^1 dx \int_0^x dy$$

$$\left[-\frac{\lambda(x-y)^2(x-y-1) \mp \{2(y-x) \pm (1-x+y)\}\, y(1-x)}{\Phi(x,y)\left\{\Phi(x,y) - y(1-x)\dfrac{\Box^2}{\varkappa_0^2}\right\}} \cdot \frac{\Box^2}{\varkappa_0^2} \right.$$

$$\left. + \frac{y(1-y+xy-x^2)}{\Phi(x,y) - y(1-x)\dfrac{\Box^2}{\varkappa_0^2}} \cdot \frac{\Box^2}{\varkappa_0^2} + (3(y-x)-1) \ln \frac{\Phi(x,y) - y(1-x)\dfrac{\Box^2}{\varkappa_0^2}}{\Phi(x,y)} \right] j_\mu(x_0), \tag{18}$$

where $\Box^2$ denotes the d'Alembertian operator, and j_μ is given by (5).

It is apparent from the expression (18) that scalar and pseudoscalar mesons have no additional magnetic moments. The same holds true for the vector and pseudovector couplings.

§ 4. Vector and pseudovector mesons. Vector and pseudovector couplings

In this case, the expression corresponding to (6) becomes

$$H=-\frac{ieg^2}{8\hbar^2c^2}A_\mu\int\int_{-\infty}^{+\infty}dx_1dx_2\,\bar{\phi}_\lambda(x_1)\phi_\nu(x_2)\;Tr\Big[S_F(x_0-x_1)\tau_P\gamma_\mu S_F(x_1-x_0)\tau_{PN}(\gamma_5)\gamma_\nu$$

$$S_F(x_1-x_2)\tau_{NP}(\gamma_5)\gamma_\lambda\Big], \tag{19}$$

where (γ_5) is unity for vector meson. As before, let

$$\phi_\nu(x)=\phi_\nu(p)\exp(ipx),\quad \bar{\phi}_\lambda(x)=\bar{\phi}_\lambda(p')\exp(-ip'x). \tag{20}$$

From the conditions

$$\frac{\partial\phi_\nu(x)}{\partial x_\nu}=\frac{\partial\bar{\phi}_\lambda(x)}{\partial x_\lambda}=0,$$

it then follows that

$$p_\nu\phi_\nu(p)=p_\lambda'\,\bar{\phi}_\lambda(p')=0. \tag{21}$$

This being taken into account, the replacement

$$p_\nu'\to(\Delta p)_\nu,\quad p_\lambda\to-(\Delta p)_\lambda \tag{22}$$

is justified when they operate on $\bar{\phi}_\lambda(p')\;\phi_\nu(p)$.

$$H=-\frac{eg^2}{4\pi^2\hbar^2c^2}A_\mu\int_0^1dx\int_0^x dy\times$$

$$\times\Big[-\frac{(p+p')_\mu}{2}\delta_{\nu\lambda}\;\frac{(1+x-y)(1-x)\Big(\frac{\Delta p}{x_0}\Big)^2}{\Phi(x,y)+\Big(\frac{\Delta p}{x_0}\Big)^2y(1-x)}-$$

$$-\frac{(p+p')_\mu}{2}\delta_{\nu\lambda}\frac{\{\lambda(x-y)^2(1-x+y)\mp(-2(x-y)\mp(1-x+y)\}y(1-x)\Big(\frac{\Delta p}{x_0}\Big)^2}{\Phi(x,y)\Big\{\Phi(x,y)+\Big(\frac{\Delta p}{x_0}\Big)y(1-x)\Big\}}$$

$$+\frac{(\Delta p)_\nu\delta_{\mu\lambda}-(\Delta p)_\lambda\delta_{\mu\nu}}{\Phi(x,y)+\Big(\frac{\Delta p}{x_0}\Big)^2y(1-x)}\Big\{\lambda(x-y)((2-x)(x-y)-1)\mp(1\pm(1-x))$$

$$+\Big(\frac{\Delta p}{x_0}\Big)^2y(1-x)^2\Big\}+\ln\frac{\Phi(x,y)+\Big(\frac{\Delta p}{x_0}\Big)y(1-x)}{\Phi(x,y)}\Big\{\frac{(p+p')_\mu}{2}\delta_{\nu\lambda}(1+x-y)-((\Delta p)_\nu$$

$$\delta_{\mu\lambda}-(\Delta p)_\lambda\delta_{\mu\nu})(2-x)\Big\}$$

$$+\frac{(p+p')_\mu(\Delta p)_\nu(\Delta p)_\lambda}{x^2}\;\frac{2\lambda y(x-y)(1-x)}{\Phi(x,y)+\Big(\frac{\Delta p}{x_0}\Big)^2y(1-x)}\Big]\bar{\phi}_\lambda(p')\phi_\nu(p)\;\exp\;(-i\Delta px_0),$$

where in the double signs the upper ones are to be taken for vector meson and the lower ones for pseudovector meson. Due to the operator interpretation of p_μ's, this reduces to

$$H=-\left(\frac{g^2}{4\pi\hbar c}\right)A_\mu\int_0^1 dx\int_0^x dy\left[Aj'_\mu(x_0)+B\frac{ie}{\hbar c}\frac{\partial}{\partial x_\nu}\left(\phi^*_\nu\phi_\mu-\phi^*_\mu\phi_\nu\right)\right.$$
$$\left.+C\frac{1}{x^2}\frac{ie}{\hbar c}\frac{\partial^2}{\partial x_\lambda\partial x_\nu}\left(\frac{\partial\phi^*_\nu}{\partial x_\mu}\phi_\lambda-\phi^*_\nu\frac{\partial\phi_\lambda}{\partial x_\mu}\right)\right], \tag{24}$$

where

$$A=\frac{1}{2}\left\{\frac{(1+x-y)\ (1-x)}{\Phi(x,y)-y(1-x)\frac{\Box^2}{x_0^2}}\right.$$
$$\left.+\frac{\{\lambda(x-y)^2(1-x+y)\mp(-2(x-y)\mp(1-x+y)\}\ y(1-x)}{\Phi(x,y)\left\{\Phi(x,y)-y(1-x)\frac{\Box^2}{x^2}\right\}}\right\}\cdot\frac{\Box^2}{x_0^2}$$

$$+\frac{1+x-y}{2}\ln\frac{\Phi(x,y)-y(1-x)\frac{\Box^2}{x_0^2}}{\Phi(x,y)}, \tag{25}$$

$$B=-\frac{\lambda(x-y)((2-x)(x-y)-1\mp(1\pm(1-x))}{\Phi(x,y)-y(1-x)\frac{\Box^2}{x_0^2}}+\frac{y(1-x)^2+\frac{1}{2}(1+x-y)(1-x)}{\Phi(x,y)-y(1-x)\frac{\Box^2}{x_0^2}}\cdot\frac{\Box^2}{x_0^2}$$

$$-\frac{1}{2}\frac{\{\lambda(x-y)^2(1-x+y)\mp(-2(x-y)\mp(1-x+y))\}y(1-x)}{\Phi(x,y)\left\{\Phi(x,y)-y(1-x)\frac{\Box^2}{x_0^2}\right\}}\cdot\frac{\Box^2}{x_0^2}$$

$$+\left(\frac{1+x-y}{2}+2-x\right)\ln\frac{\Phi(x,y)-y(1-x)\frac{\Box^2}{x_0^2}}{\Phi(x,y)}, \tag{26}$$

$$C=-\frac{2\lambda y(1-x)\ (x-y)}{\Phi(x,y)-y(1-x)\frac{\Box^2}{x_0^2}}. \tag{27}$$

While the second term of (24) gives, according to Pauli[3], the anomalous magnetic moment of meson, the third term is due to an intrinsic quadrupole moment as pointed out by Kinoshita and Nambu[1].

Thus, the magnetic moments in meson magnetons $e/2x$ are:

$$\text{Vector meson}: \quad \mu_v=-\left(\frac{g^2}{4\pi\hbar c}\right)\frac{1}{\pi}\int_0^1 dx\int_0^x dy\,\frac{\lambda(x-y)\left((2-x)(x-y)-1\right)+x-2}{\Phi(x,y)}, \tag{28}$$

Pseudovector meson:

$$\mu_{pv}=-\left(\frac{g^2}{4\pi\hbar c}\right)\frac{1}{\pi}\int_0^1 dx\int_0^x dy\,\frac{\lambda(x-y)\,((2-x)\,(x-y)-1)+x}{\Phi(x,y)}. \tag{29}$$

Performing the integration and inserting the value of π meson mass 278m, and therefore $\lambda=0.023$ (which is very small and hardly contributes to the evaluation), we get

$$\mu_v=+0.21\left(\frac{g^2}{4\pi\hbar c}\right),\ \mu_{pv}=-0.11\left(\frac{g^2}{4\pi\hbar c}\right). \tag{30}$$

On the other hand, for the quantity measuring the magnitude of the quadrupole moment

$$Q=\left(\frac{g^2}{4\pi\hbar c}\right)\frac{1}{\varkappa^2}\int_0^1 dx\int_0^x dy\,C \tag{31}$$

we obtain the value

$$Q=-0.0001\frac{1}{\varkappa^2}\left(\frac{g^2}{4\pi\hbar c}\right)=-1.9\times10^{-30}\left(\frac{g^2}{4\pi\hbar c}\right)\text{cm}^2. \tag{32}$$

It is to be noted that Q has the same value both for vector and pseudovector mesons.

§ 5. Tensor and pseudotensor couplings

While in the case of nucleon, the calculation for tensor coupling gave a logarithmic divergent expression, a similar though rather complicated procedure of calculation yielded a definite result in our case.

To quote only the main results attained, the magnetic moment and the quantity Q are given by

$$\left(\frac{g'^2}{4\pi\hbar c}\right)\frac{1}{\pi}\int_0^1 dx\int_0^x dy\,\frac{\lambda(x-y)\{x(x-y)-1\}+(1-x+y)\mp(-1\mp y)}{\Phi(x,y)}, \tag{33}$$

and

$$\left(\frac{g'^2}{4\pi\hbar c}\right)\frac{1}{2\pi}\int_0^1 dx\int_0^x dy\,\frac{\lambda(x^2-5y^2-3x-y+2)\,(x-y)\mp\{2(1-x+y)\pm(1-x-y)\}}{\Phi(x,y)}, \tag{34}$$

respectively. The evaluation gives

$$\mu_v=+0.21\left(\frac{g'^2}{4\pi\hbar c}\right),\quad \mu_{pv}=-0.11\left(\frac{g'^2}{4\pi\hbar c}\right), \tag{35}$$

and

$$Q_v=-Q_{pv}=-0.11\frac{1}{\varkappa^2}\left(\frac{g'^2}{4\pi\hbar c}\right)=-2.1\times10^{-27}\left(\frac{g'^2}{4\pi\hbar c}\right)\text{cm}^2. \tag{36}$$

§ 6. Conclusion

Using the method of Case, corrections to the meson current due to interaction with the nucleon field have been calculated up to the second order for all types of mesons and couplings. Definite results for the additional magnetic moments of the vector and pseudovector mesons have been obtained. Their values are at most several percent of the normal values. There also appeared for the case of vector and pseudovector mesons the terms representing an intrinsic quadrupole moment.*) For vector and pseudovector couplings, its value is very small and the sign is the same for both types of meson. For tensor and pseudotensor couplings, it has the magnitude which is no longer negligible, and, moreover, its sign is opposite to each other for the two types of meson. These situations might have an role, in future, in deciding the type of π meson.

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*) It is intended to give the discussion about the quadrupole moment in a separate paper.

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Effect of Nuclear Binding on Meson Production

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§ 1. Introduction

Recently, various experiments on the artificial meson production were carried out at Berkeley[1]. It seems possible to derive some conclusion on the meson from their data and the theoretical calculations performed by many authors[2],[3]. But when we are going to compare both results, we at once encounter many difficulties, among which the effect of nuclear binding is the most important. We must not forget that the experiments are performed by bombarding high energy particles on the graphite target in most cases, whereas the calculations are done only for elementary processes. Therefore, it is necessary to estimate at first this effect of nuclear binding. Here, we calculate it assuming the Fermi gas model for a nucleus, and furthermore, other effects will be also taken into account, if they are necessary.

After some preliminary discussions (§ 2), we treat the case of X-ray bombardment (§ 3) and of proton bombardment (§ 4), separately. The energy spectrum, the excitation function and the angular distribution of produced mesons are calculated. Also we determine the coupling constant of the nucleon-meson interaction, by comparing theoretical results with experimental ones. Finally we discuss the results, which seem to favour the pseudo-scalar meson theory. If this is the case, higher order calculations will be necessitated.

§ 2. Qualitative discussions

It is well known that the Fermi gas model for a nucleus is very useful to explain high energy nuclear reactions[4]. If we adopt this model, we see that there are two kinds of effects of nuclear binding on meson production: 1) the spread of available energy for meson production and 2) the restriction of possible final states.

Let us discuss here these effects briefly.

Effect 1) The momentum of a nucleon inside the nucleus (target nucleon) is $\leq P_F$ (P_F=Fermi momentum, $E_F=P_F^2/2M$=Fermi energy and M=nucleon mass).

Thus the spread of available energies of incident γ-rays or protons due to this internal motion of a target nucleon amounts to $\leq \frac{kP_F}{Mc} = 2\sqrt{kE_F}\sqrt{\frac{k}{2Mc^2}}$ or $\leq 2\sqrt{EE_F}$, where k or E means the initial energy of incident γ-rays or protons, respectively. In the special case, where $k=E$, we see that this effect is more important for the proton bombardment than for the γ-ray bombardment.

Effect 2). This effect reduces the number of final states, since the Pauli principle requires that the final momenta of nucleons in laboratory system must be larger than P_F. The momentum of incident γ-ray or proton is k/c or $P=E\sqrt{\frac{2M}{E}}$, respectively. P is considerably larger than k/c for $k=E$. Therefore, we must suppose that more final states are excluded in the case of γ-ray bombardment than the proton case.

As has already been pointed out, the effect 1) reduces the threshold energy for the meson production considerably[5]. In general, we may consider that the effect 1) increases the cross-section for meson production. While the effect 2) forbids some final states and makes the cross-section smaller.

The above estimation shows that the effect of nuclear binding may be quite different for the different kinds of incident particles. In the case of γ-ray bombardment, the effect 2) will predominate and will make the cross-section smaller. On the contrary, the effect 1) will be more important in the proton case, and will make the cross-section larger.

§ 3. Meson production by x-rays

In the experiment of McMillan et al.[6], they bombarded the graphite cylinder by X-rays with the maximum energy 335 Mev, and detected mesons which were emitted nearly at the right angle to the incident beam. For analysis of their data, we must take into account following corrections besides the above effects of nuclear binding.

3) Energy spectrum of X-rays
4) Geometry of experimental arrangements
5) Scattering from the absorber
6) Absorption in the carbon cylinder.

We distinguish the laboratory system (the target nucleus is at rest) from the nucleon system, in which a nucleon is initially at rest As is easily seen, the changes in momenta of a meson and a γ-ray, arising from the transformation from the nucleon system to the laboratory system are negligible, while that for nucleon is of a significant order of quantity. This allows us to neglect the effect 1) and to calculate the energy and the intensity of mesons in the nucleon system.

Then the effect 2), i.e. the exclusion of final states, can be easily estimated as follows. First we determine the recoil momentum of a nucleon in the nucleon

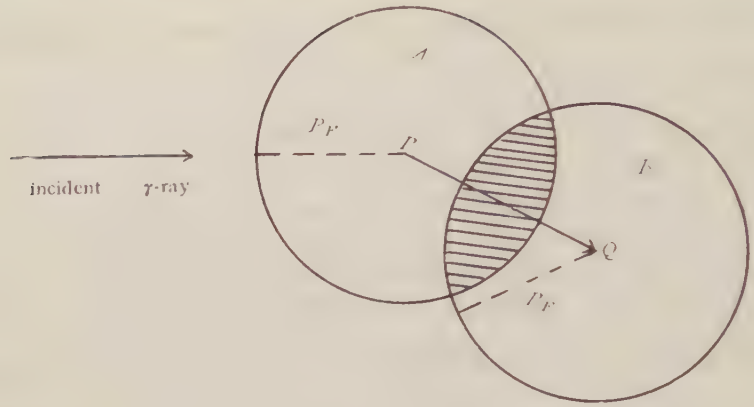


Fig. 1. $\overrightarrow{PQ}$ is the recoil momentum in the nucleon system. Let P be the origin of the momentum space in the laboratory system. Then the initial momentum of nucleon is uniformly distributed in the sphere A, and the final recoil momentum in B. Pauli's principle excludes the final states corresponding to the intersection of A and B, which is shown as the hatched part.

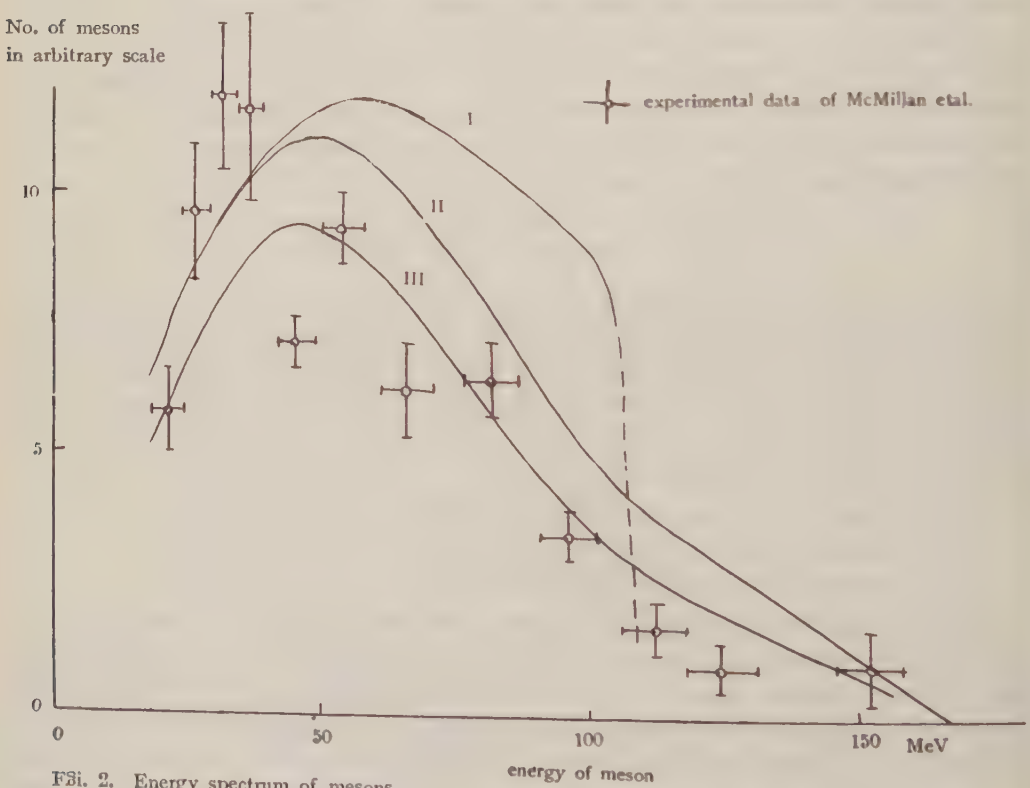


Fig. 2. Energy spectrum of mesons
a) Scalar meson with scalar coupling.
Curve I: We consider only 2) and 3).
Curve II: Furthermore, effect 1) and 4) are added.
Curve III This final curve is obtained from II with corrections 5) and 6).
Theoretical and experimental curves are normalized to fit both best.

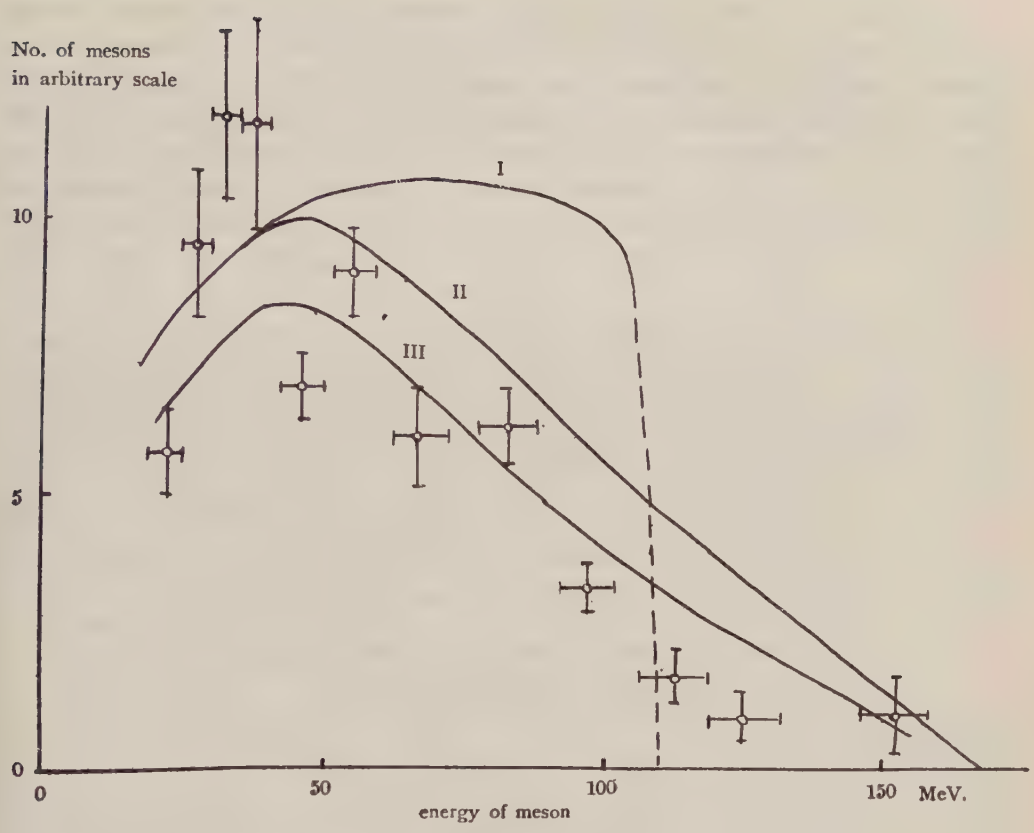


Fig. 2. b) Pseudoscalar meson with pseudoscalar coupling.

system, which is a function of the incident momentum of γ-ray and also of the angle or energy of the emitted meson. By the transformation into laboratory system, this recoil momentum is distributed uniformly in the sphere B (Fig. 1). Then we can find the forbidden part of final states as the intersection of the two spheres A and B.

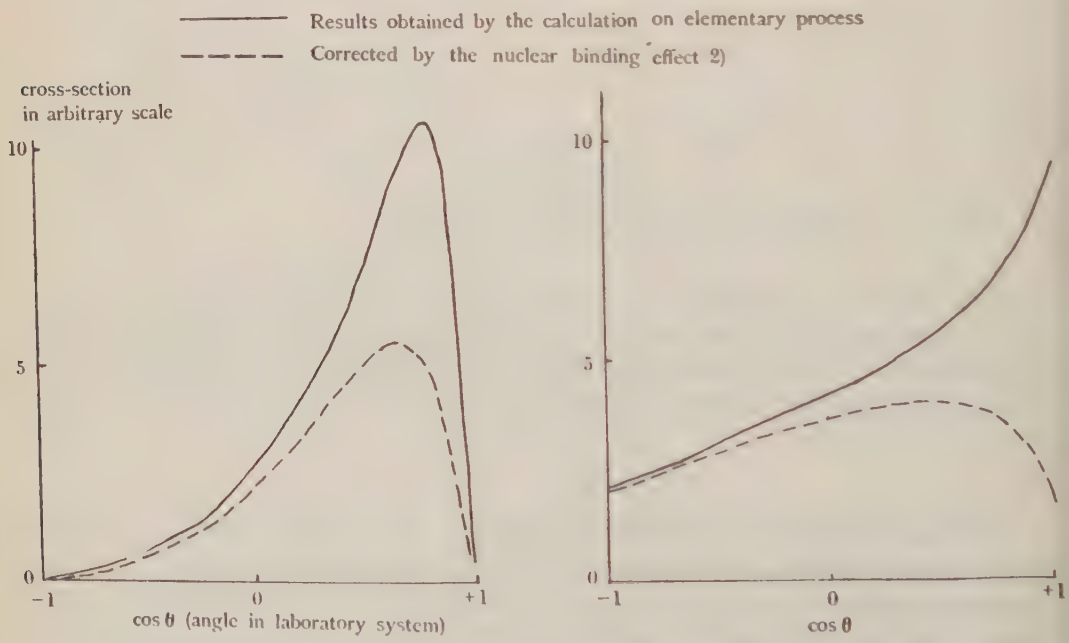
Energy spectrum of produced mesons We consider the case of the scalar meson with scalar coupling, $S(s)$, and the pseudoscalar meson with pseudoscalar coupling, $Ps(ps)$. The cross-sections of meson production for these cases have been calculated by various authors[2].

The curve I in Fig. 2 shows the energy spectrum of mesons emitted at the right angle to the incident beam, where only the effect 2) and 3) are taken into account. As seen at once, this curve is nearly flat and drops sharply at $\sim$ 110 Mev, whereas the experimental one has a long tail in higher energy regions. This shows that further corrections are necessary.

Heretofore, we have approximated the energy of mesons as uniquely determined by the incident energy of γ-rays. The effect 1) and 4) make the energy of mesons distribute around the above unique value. We have calculated this energy distribution of mesons for a fixed energy of γ-rays and given it in the appendix. The half-width of this distribution is an increasing function of the incident energy of γ-rays. Therefore, it may be sufficient to take it into account only in the region of higher energy, rather at the upper end of the energy spectrum, where it drops sharply (curve I in Fig. 2). The curve II in Fig. 2 is obtained in this way.

Furthermore, the corrections 5) and 6), i.e. scattering out and absorption, are estimated. For the former, we apply the formula of multiple scattering[7], which reduces the intensity of mesons in high energy region. For the latter, we take into account only the ionization loss[8], which is effective in lower energy region.

The curve III in Fig. 2 includes all corrections estimated above. We find that there is no appreciable difference between the cases $S(s)$ and $Ps(\not{p}s)$, and both are nearly in agreement with the experiments.



i) Scalar meson with scalar coupling

ii) Pseudoscalar meson with pseudoscalar coupling

Fig. 3. Angular distribution of mesons produced by γ-ray with energy 230 Mev.

Angular distribution McMillan et al. found experimentally that the angular distribution of mesons is nearly isotropic, if the geometry of their experi-

mental arrangement, i.e. the effect 4), is taken into account. The experimental data on mesons with energies 43～59 Mev. are shown in their paper[6].

We calculate the angular distribution of mesons produced by monochromatic γ-rays with energy 230 Mev. Our results may be compared with their experimental data, since this γ-ray emits a meson with energy～50 Mev. at the right angle to its direction.

The full curve in Fig. 3 is the angular distribution obtained by the calculation on the elementary process[2]. In both cases $S(s)$ and $P_s(ps)$, the pronounced maximum is seen in the forward direction.

The effect 2) considerably reduces this maxima, since the meson emitted in the forward direction is accompanied by the small recoil nucleons and large part of the final states are excluded. We represent in Fig. 4 this correction factor for γ-ray with various energies. Applying this factor to the above full curve in Fig. 3, we get the corrected angular distribution for 230 Mev. γ-rays. This is shown by the broken curve in Fig. 3.

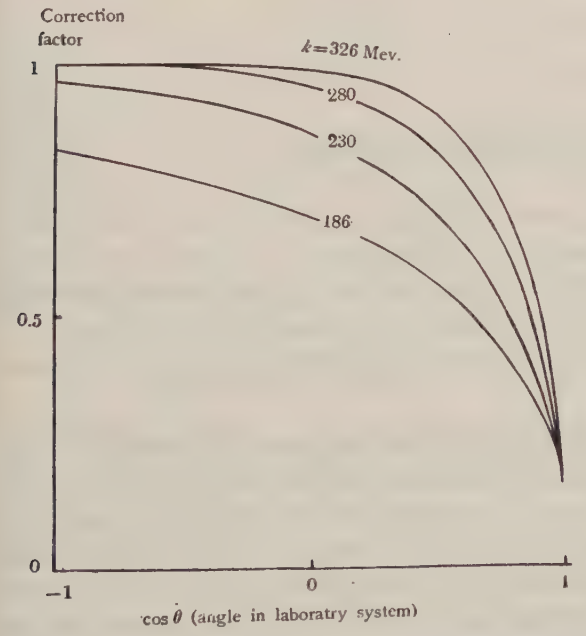


Fig. 4. Correction factor due to nuclear binding on the angular distribution of mesons produced by monochromatic gamma-rays.

Further corrections are not important. We may conclude that the angular distribution is nearly isotropic for the case $P_s(ps)$. In the case $S(s)$ the strong maximum in the forward direction still remains and it seems to contradict with the experiment.

Absolute yield Finally, we compare the theoretical and experimental absolute intensity of mesons emitted at the right angle to the incident beam. The mean cross-section per one quantum and one nucleon for meson production by X-rays with energies 140～335 Mev. is determined from the experiments as

follows,

$$3.8\times10^{-30}\ \text{cm}^2/\text{sterad}.$$

The scattering out and the absorption reduce the meson yield approximately by 20%. Thus the corrected value is 4.8×10^{-30} cm^2/sterad. While the theoretical values are

$$\frac{e^2}{\hbar c}\frac{f^2}{\hbar c}\times5.5\times10^{-28}\ \text{cm}^2/\text{sterad, for } S(s),$$

and

$$\frac{e^2}{\hbar c}\frac{f^2}{\hbar c}\times3.2\times10^{-29}\ \text{cm}^2/\text{sterad, for } Ps(ps).$$

The effect 2) makes the cross-section nearly 15% smaller than that in free collisions. Comparing these figures each other, we can determine the coupling constant f of the nucleon-meson interaction as follows:

$$f^2/\hbar c=1.15 \qquad \text{for } S(s),$$

and

$$f^2/\hbar c=16 \qquad \text{for } Ps(ps).$$

§ 4. Meson production by protons

Next we consider the case of proton bombardment, in which we need only to take account of the effects 1) and 2) described in § 2. Here we use the center of mass system of the incident proton and one of the target nucleons, because theoretical cross-sections are conveniently expressed in this system.

First we treat the effect 1), for it is more important than the effect 2), which will be discussed later.

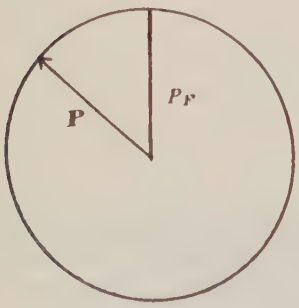


Fig. 5. The momentum $\boldsymbol{p}$ of a nucleon in a target nucleus is assumed to be distributed uniformly on the surface of the sphere with radius p_F.

Energy spectrum of produced mesons[1] For simplicity, we assume that the momentum $\boldsymbol{P}$ of a nucleon in a target nucleus is distributed uniformly on the surface of the occupied sphere with radius $P_F(\approx200$ Mev./$c)$ in momentum space (see Fig. 5). The energy spectrum of produced mesons in the center of mass system have been given by Fukuda and Takeda[3]. Transforming their result into the laboratory system (the target nucleus is at rest), multiplying the suitable weight factor, which is proportional to $1-\dfrac{\boldsymbol{P}_0\boldsymbol{P}}{P_0P_F}$, and finally summing up over all possible orientations of $\boldsymbol{P}$, we get the spectrum which can be directly compared with the observed

data[1] (*cf.* Fig. 5). The full curve in Fig. 6 shows the energy spectrum of mesons produced by 345 Mev. protons for the case of $Ps(ps)$. On the other hand, if we do not take into account this effect, we get the result expressed by the dotted curve in Fig. 6, which is not in agreement with the observed results. Furthermore, this effect leads to a similar spectrum of mesons even if we assume $S(s)$. Thus we see that the main feature of the observed energy spectrum can be explained by the effect 1) and the assumed type of mesons is not relevant.

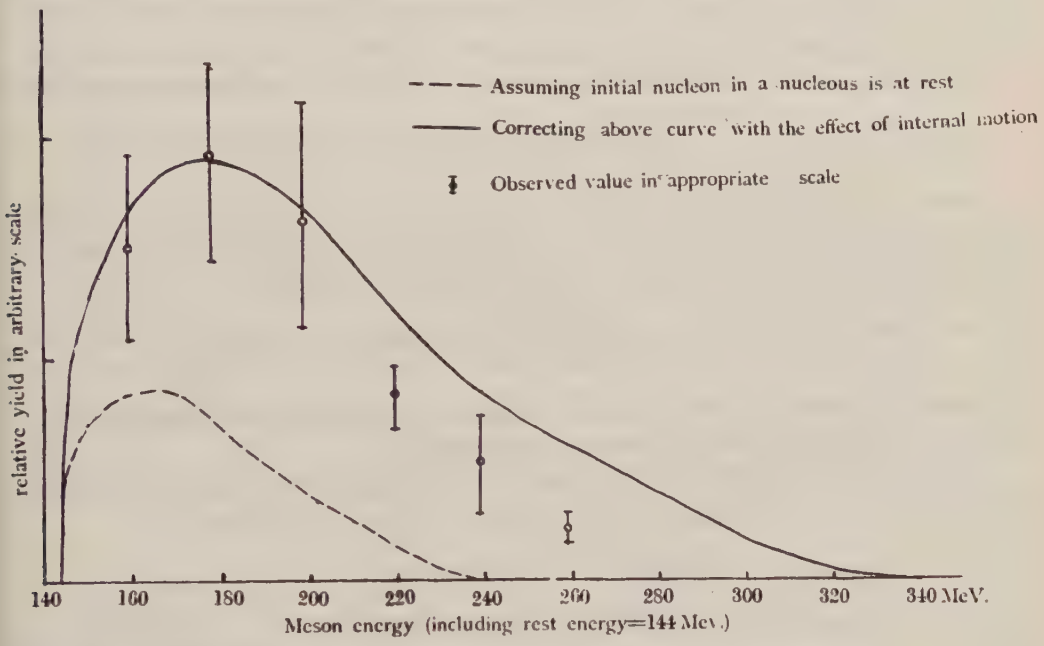


Fig. 6. Energy spectrum of mesons (π^+ and π^-) (345 Mev. proton bombardment)

Absolute yield[1] The total cross-section per one target nucleon for 345 Mev. protons can be obtained by integrating the above result over the meson energy.

(1) For $Ps(ps)$:

$$\sigma=\left(\frac{f^2}{\hbar c}\right)^3\left(\frac{\hbar}{Mc}\right)^2(0.0318\ a-0.00113\ \beta),\qquad r\approx 5.$$

(2) Assuming the pseudo-scalar meson theory with pseudo-scalar coupling and using the phenomenological nuclear force[3] (quoted as $Ps(ps,\ N.\ F.)$),

$$\sigma=\frac{f^2}{\hbar c}\left(\frac{\hbar}{Mc}\right)^2\times 9.5a\times 10^{-4},\qquad r\approx 3.$$

Where a and β are listed in Table I.

r means the ratio of the number of positive mesons to that of negative mesons, and the observed value[1] is $r\approx 3$. In the above calculations numbers of protons

Table I.

	α	β
$pp \to pp\pi^0$	0	0
$pn\pi^+$	2	−2
$pn \to pp\pi^-$	1	1
$pn\pi^0$	2	0
$nn\pi^+$	1	1
$nn \to pn\pi^-$	2	−2
$nn\pi^0$	0	0

In this table, p, n, π^0, π^+, and π^- mean proton, neutron, and neutral, positive, and negative meson, respectively.

and neutrons in the target nucleus have been assumed to be equal. For comparison, we quote here the result for the case where the initial target nucleon is at rest and the nuclear binding is not taken into account.

(3) For $Ps(ps)$; $$\sigma_f=\left(\frac{f^2}{\hbar c}\right)^3\left(\frac{\hbar}{Mc}\right)^2(0.0102\alpha-0.0044\ \beta), \qquad r\approx 6.$$

Comparing (1) with (3), we see again that the effect of nuclear binding 1) is quite important in the case of proton bombardment.

Equating these results to the observed total cross-section per one nucleon 2.7×10^{-27} cm^2, we can determine the coupling constant as follows;

$$f^2/\hbar c=2.0 \qquad \text{for } Ps\ (ps),$$

$$f^2/\hbar c=270 \qquad \text{for } Ps\ (ps,\ N.F.),$$

and

$$f^2/\hbar c=0.35 \qquad \text{for } S(s).$$

The results of $Ps(ps)$ and $S(s)$ seem reasonable, while $Ps,(ps,\ N.F.)$ leads to quite an unreasonable result. This may be due to the special character of the phenomenological force used here. Therefore, we may suppose that this assumed phenomenological nuclear force is not adequate in higher energy region.

Furthermore, if we assume that the momentum distribution of target nucleons is uniform inside the occupied sphere, the agreement of the calculated and the observed energy spectrum must be better than in Fig. 6.

In order to examine the effect 2), i.e. exclusion of final states, we consider the two typical cases (a) and (b), as given in Fig. 7. In the former case (a), the final nucleons have relatively large momenta because the resultant momentum of the two initial nucleons is large. And the effect 2) is rather small. On the contrary, the final nucleons in the latter case (b), may have smaller momenta and thus considerable effect is expected. However the contribution of the case

(b) to the total cross-section can not be so large. Thus we may neglect the effect 2) in our calculations. Strictly speaking, the consideration of the effect 2) results in the decrease of the number of mesons with higher energies, since these mesons are produced mainly in the case (b). This also makes the agreement of the calculated and experimental energy distribution better than in Fig. 6.

Excitation function for slow meson production[1] Let $\sigma(E,\varepsilon)$ denote the cross-section for production of meson with energy ε by an incident proton with energy E in the nucleon system defined in § 3. Then, taking into account the effect 1), the yield of a meson with energy ε is approximately given by the following expression:

$$\bar{\sigma}(E,\varepsilon)=\int_{-P_F}^{P_F}\frac{P+P_F}{P}\sigma\left(\frac{P^2}{2M}+\frac{PP'}{M},\ \varepsilon\right)\frac{3(P_F^2-P'^2)}{4P_F^3}\,dP',$$

where P means the momentum of incident proton. Fig. 8 shows the relative value of $\sigma(E,\varepsilon)$ for $\varepsilon=6$ Mev. in the case $Ps(ps)$, while the observed results represent the relative yield of π^--mesons with energies 2—10 Mev.[1] The agreement of both results seems satisfactory. This excitation function depends mainly on the available volume in momentum space, for which $\frac{P^2}{2M}+\frac{PP'}{M}$ is larger than the threshold energy. And the assumed type of mesons is not essential.

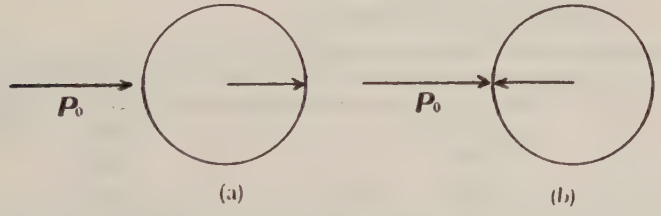


Fig. 7

§ 5. Conclusions

From the above analysis, we see that the effect of nuclear binding is quite different for different kinds of bombarding particles. Furthermore, even if we consider the mesons produced by one kind of particles, this plays a significant role for some of their behaviour, while only a little for the other behaviours. Thus it is desirable to compare such experimental and theoretical results as are not seriously affected by the nuclear binding, in order to determine the type and the

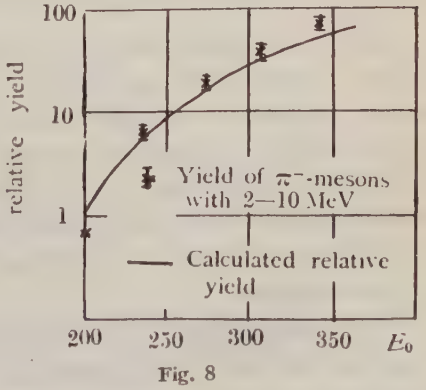


Fig. 8

coupling constant of the mesonic interaction.

The above analysis are summarized as follows:

1) We can say only little about the type of mesons from their energy spectrum, because the spectrum is largely modified by the nuclear binding from its original form, which is obtained through the calculation on the elementary process. The modified spectrum is nearly in agreement with the experimental one, whether we assume $S(s)$ or $Ps(ps)$.

2) The angular distribution of mesons produced by X-rays are of special interest. The experimental data seem to favour the $Ps(ps)$.

3) The excitation function for mesons produced by the proton beam depends only a little on the assumed type of mesons. The experimental curve can be explained for the most part by the available volume in momentum space, and the cross-section plays only a little role there.

4) The ratio of positive to negative mesons with high energies is little affected by the nuclear binding. But this ratio depends so slightly on the type of mesons that it is hard to deduce any conclusion at all. More precise experiments are hoped on this point.

5) At present the absolute yield of mesons are the most important data. We can determine the coupling constants from them. Our results are summarized in Table II.

Table II.

The values of coupling constants

Determined from	meson production by X-rays	meson production by protons
$S(s)$	1.15	0.35
$Ps(pv)$	0.1	0.2
$Ps(ps)$	16	2.0
$Ps(ps, N.F.)$	–	270

As is seen from this table, $Ps(ps)$ gives inconsistent values for the coupling constant, while $S(s)$ and $Ps(pv)$ lead to somewhat satisfactory ones. But it may be possible that the 4-th order calculation revises this inconsistency in the case of $Ps(ps)$[9]. Furthermore, the phenomenological nuclear forces fitted to the lower energy phenomena do not seem adequate in higher energy regions. The same conclusion was also derived from analysis of cosmic-ray stars[10]. In connection with capture processes of negative mesons, further discussion will be given in another paper.

Acknowledgement: Authors wish to express their sincere thanks to Prof. Taketani, Messrs. S. Hayakawa and G. Takeda for their kind interest taken in this work.

Appendix

Distribution in meson energy due to effects 1) and 4). Let $\boldsymbol{k}_0$ and $\boldsymbol{P}_0$ be the incident momentum of γ-ray and nucleon and $\boldsymbol{q}$ and $\boldsymbol{P}$ be the final momentum of meson and nucleon, in the laboratory system. We may consider $\boldsymbol{P}_0$ only in the direction of $\boldsymbol{k}_0$, for all are symmetric with respect to this direction. The kinetic energy E_q of meson is obtained from the conservation of energy and momentum.

$$E_q=E_0-\frac{k_0P_0}{M}+\frac{(k_0+P_0)q\ \cos\theta}{M}, \tag{1}$$

where θ is the angle between $\boldsymbol{k}_0$ and $\boldsymbol{p}$, and E_0 is the value of Eq for $\theta=\frac{\pi}{2}$, $P_0=0$ $(P_0=(\boldsymbol{k}_0\boldsymbol{P}_0)/k_0)$.

The distribution function of P_0 is given by

$$\frac{3}{4}(1-y^2)dy \qquad (1\geq y\geq -1), \tag{2}$$

where

$$y=P_0/P_F.$$

From the experimental arrangement, the detection probability of mesons emitted with an angle θ is approximately given by,

$$\frac{3}{4}(1-x^2)dx \qquad (1\geq x\geq -1), \tag{3}$$

where

$$x=\cos\ \theta.$$

E_q can be expressed by new variables x and y as follows,

$$E_q=E_0+Ax-By+Cxy, \tag{4}$$

$$A=\frac{k_0q}{M}, \quad B=\frac{k_0P_F}{M} \quad \text{and} \quad C=\frac{P_Fq}{M}.$$

q in B and C may be approximated by $\sqrt{E_0^2+2E_0\mu}$, i.e. the value at $\theta=\frac{\pi}{2}$, $P_0=0$. Then we have following numerical values of A, B and C for $k_0=340$ Mev

$$A=77 \text{ Mev.}, \quad B=75 \text{ Mev.} \quad \text{and} \quad C=45 \text{ Mev.}$$

We may neglect the term Cxy in (4), as we are interested in the case for which $xy\ll 1$, and as C is somewhat smaller than A and B. Then it becomes simply

$$E_q=E_0+a(x-y), \tag{5}$$

$$a \approx 75 \text{ Mev.}$$

The probability $F(E)$ for E_q to be larger than E is obtained by the integral,

$$F(E)=\frac{9}{16}\iint_{\Sigma} dx\ dy\ (1-x^2)(1-y^2). \tag{6}$$

The integration is carried out through the region Σ, as shown in Fig. 9. Neglecting the higher order terms in $\frac{E-E_0}{a}$, we obtain

$$F(E)=\frac{1}{2}+\frac{3}{5}\frac{E-E_0}{a}+0\left(\frac{E-E_0}{a}\right)^3. \tag{7}$$

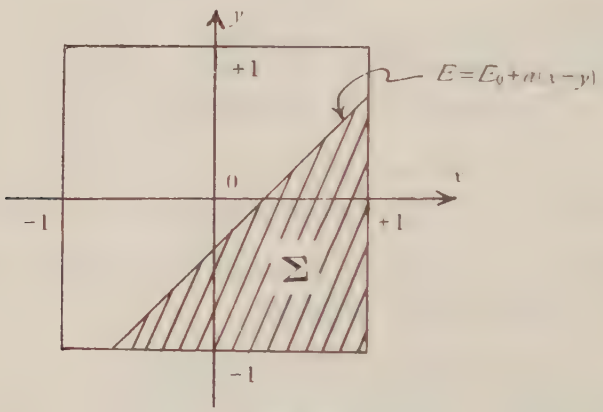


Fig. 9.

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On the Interaction of Mesons and Nucleons*

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(Receieved August 5, 1950)

§ 1. Introduction and Summary

Yukawa's meson theory received brilliant support in the recent experiments concerning mesons artificially produced by the Berkley cyclotron.

It is, however, well known that in spite of the advance of the quantum electrodynamics and qualitative agreements between theoretical results and experimental data, the problem of nuclear forces and nuclear magnetic moments have not been solved completely. These facts show that the present model of π meson is inadequate to describe those phenomena. Now let us consider the history of the meson types. After the prediction of Yukawa, Kemmer[1] found that there exist four different kinds of meson fields. The development of the meson theory based on his classification seemed to be successful in the qualitative interpretation of almost all phenomena.

Especially, the pseudoscalar meson field has been studied by several authors. Further, the mixed field theories came out in order to remove the $1/r^3$ singularity by compensation. However, these theories depend on the Kemmer's assumption. By the appearance of Tomonaga-Schwinger theory, which is based on the idea of renormalization of mass and charge, ambiguities which remained in the subtraction procedure are removed. In this way, as Case showed, it turned out to be possible to obtain the finite results for the nuclear magnetic moments in the scalar and pseudoscalar fields which were not in accord with experiences. The vector and pseudovector fields give divergent results owing to their tensor couplings.[2]

Under these circumstances, in order to remedy the difficulties, we investigated[3] in the previous note the generalization of the interaction Lagrangians suggested by Schoenberg,[4] and obtained nuclear potentials for the spin 0 field, though no explanation for the introduction of those interactions was given. Further, magnetic moments of nucleons by the spin 0 field were calculated according to Feynman-Dyson's formalisum (unpublished).

It was shown that the contributions to the magnetic moments by generalizing

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* H. Enatsu, Prog. Theor. Phys. **5** (1950), 102. This paper will be referred to as I.

interactions are of small influence on the results. That is, the usual terms have the form

$$(\text{const.})\,(\psi^*\rho_3\boldsymbol{\sigma}\boldsymbol{H}\psi),$$

on the other hand, the contribution from the current which comes from the process that a nucleon emits a meson by the scalar coupling and reabsorbs it by the pseudoscalar coupling, is given as follows:

$$(\text{const.})\,(\psi^*\rho_2\boldsymbol{\sigma}\boldsymbol{H}\psi).$$

It is clear that the latter is not large compared with the former due to its operator ρ_2. Such an attempt carried out hitherto, has led to the conclusion that we could not expect the quantitative agreement for the spin 0 field.

At the present stage of our knowledge it is of interest to investigate further the spin 1 field theory. Contrary to the spin 0 field, it has the analogy with the electromagnetic field. It is the aim of this paper to draw attention to a classical aspect of the spin 1 field, and to show more insight into the meaning of the generalization of interaction. It will be shown that our basic ideas are expressed as follows,

Maxwell equations → Extended Maxwell equations by Dirac,
↓ ↕
Vector, Pseudovector } field equations → Spin 1 meson field equations.

In §2 we shall briefly survey the extended Maxwell equations by Dirac.[5] He put forward the question with respect to the dissymmetry of the Maxwell equations between the electric charge and magnetic pole, and showed that, although there is the symmetry between them, the great difference between the numerical values of them explains why electric charges are easily produced and not magnetic poles. We shall examine the formal extension of this theory for the classical vector meson field, and find the same relation as before. In §3, taking account of the dual charactor of fields, Kemmer's vector and pseudovector fields are re-examined, and it is found that there is a hope that a new point of view will lead to an idea for modification of Kemmer's theory. Nuclear forces are derived for the spin 1 field according to a canonical transformation method in §4. Then concluding remarks on the results are given in the final section.

§2. Theory of Maxwell and its extension by Dirac

Now, briefly, we shall sketch Dirac's theory. His relativistic notations are used throughout this section. (See reference (5)).

The ordinary Maxwell equations are

$$\begin{cases} \dfrac{\partial F_{\mu\nu}}{\partial x_\nu} = -4\pi j_\mu & (1) \\ \dfrac{\partial (F^\dagger)_{\mu\nu}}{\partial x_\nu} = 0\,, & (2) \end{cases}$$

where $F_{\mu\nu} = -F_{\nu\mu}$, $(F^\dagger)_{01} = F_{23}$ and $(F^\dagger)_{23} = -F_{01}$.
Dirac's extensions are

$$\begin{cases} \dfrac{\partial F_{\mu\nu}}{\partial x_\nu} = -4\pi j_\mu = -4\pi \sum_e e \int \left(\dfrac{dz_\mu}{ds}\right) \delta_4(x-z)\,ds\,, & (3) \\ \dfrac{\partial (F^\dagger)_{\mu\nu}}{\partial x_\nu} = -4\pi k_\mu = -4\pi \sum_g g \int \left(\dfrac{dz_\mu}{ds}\right) \delta_4(x-z)\,ds\,. & (4) \end{cases}$$

The last equation asserts that the modified theory allows magnetic poles. It is, however, remarkable that equation (4) is reduced to the following form

$$\frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\mu\nu}}{\partial x^\lambda} = -4\pi \tilde{k}_{\lambda\mu\nu}\,, \tag{5}$$

namely, in the usual notation (not the dual one, $F^\dagger_{\mu\nu}$) equation (5) shows that the current of magnetic poles is a pseudovector.

The usual introduction of electromagnetic potentials,

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\,, \tag{6}$$

is no longer possible when there are magnetic poles, therefore, equation (6) must be replaced by an equation of the form

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + 4\pi \sum_g (G^\dagger)_{\mu\nu}\,, \tag{7}$$

where

$$G_{\mu\nu} = g \iint \left(\frac{\partial y_\mu}{\partial \tau_0} \frac{\partial y_\nu}{\partial \tau_1} - \frac{\partial y_\mu}{\partial \tau_1} \frac{\partial y_\nu}{\partial \tau_0} \right) \delta_4(x-y)\, d\tau_0 d\tau_1\,, \tag{8}$$

$$\frac{\partial G_{\mu\nu}}{\partial x_\nu} = g \int \left(\frac{dz_\mu}{ds}\right) \delta_4(x-z)\,ds\,, \tag{9}$$

and y_μ is a general point on the two-dimensional sheet in space-time which is generated by Dirac's string.

We recall here the expression of the tensor coupling in the vector meson theory. The last term of equation (7) represents the source functions which correspond to the electric and magnetic dipoles in the meson theory.

The retarded potentials due to a charged particle and a pole are found to be given by the expressions

$$A^*_\nu(x) = e \int_{-\infty}^{\infty} J(x-z) \left(\frac{dz_\nu}{ds}\right) ds\,, \tag{10}$$

and

$$A_\nu^*(x)=g\epsilon_{\nu\lambda\rho\sigma}\iint \frac{\partial y^\lambda}{\partial \tau_0}\frac{\partial y^\rho}{\partial \tau_1}\frac{\partial J(x-y)}{\partial x_\sigma}d\tau_0 d\tau_1 , \tag{11}$$

where

$$\left.\begin{aligned} J(x)&=2\delta(x_\mu x^\mu) \qquad \text{for} \quad x_0>0 \\ &=0 \qquad\qquad\quad \text{for} \quad x_0<0. \end{aligned}\right\} \tag{12}$$

and $\epsilon_{\nu\lambda\rho\sigma}$ is the antisymmetric tensor of fourth rank with $\epsilon_{0123}=1$ as defined in (I). It will be seen that potentials and dual fields satisfy the following relations,

$$(F^\dagger)^{\alpha\beta}=\epsilon^{\mu\nu\alpha\beta}\frac{\partial A_\nu^*}{\partial x^\mu}-4\pi G^{\alpha\beta} , \tag{13}$$

$$\Box A_\mu^*(x)=4\pi\sum_e e\int\left(\frac{dz_\mu}{ds}\right)\delta_4(x-z)ds+4\pi\sum_g\frac{\partial(G^\dagger)_{\mu\nu}}{\partial x_\nu} . \tag{14}$$

Finally, Dirac's conclusion with respect to the coupling constants is as follows: All charges should be integral multiples of a unit charge e connected with the pole strength g by the relation

$$eg=\frac{1}{2}\hbar c . \tag{15}$$

As is well known, the fine structure constant is

$$\frac{e^2}{\hbar c}=\frac{1}{137} , \tag{16}$$

then, we obtain the value of g,

$$\frac{g^2}{\hbar c}=\frac{137}{4} . \tag{17}$$

"The great difference between e and g explains why electric charges are produced and not magnetic poles," he stated. This is the outline of the extended Maxwell theory by Dirac.

We shall now return to the meson theory, and attempt to test the Dirac's idea. We may summarize here the main points which are necessary to the later discussions.

(1). The formal extension of Dirac's theory for the neutral vector meson field (Proca's field), is possible, and between a mesic charge f_1 and a mesic pole f_2 there exists the similar relation that holds in his theory. In our cases a term $\varkappa^2 A_\mu(x)$ must be added to the left-hand side of equations (1), (3) and (14), and e and g are replaced by f_1 and f_2 respectively. Further, retarded potentials and retarded fields are modified in order to include the meson mass effect. However, their explicit expressions are not necessary to find the relation of f_1 and f_2 in our rough discussions, therefore for simplicity we shall discard them entirely. Thus

we have for a small meson mass ($\varkappa \sim 0$),

$$f_1 f_2 \sim \frac{1}{2}\hbar c\,, \tag{18}$$

or

$$\left(\frac{f_1^2}{\hbar c}\right)\cdot\left(\frac{f_2^2}{\hbar c}\right) \sim \frac{1}{4}\,. \tag{19}$$

The numerical values are appreciated, i.e.

$$\begin{aligned} &\left(\frac{f_i^2}{\hbar c}\right) \sim 0.5,\ \ 1\ \ ,\ \ 2\ \ \ ,\ \ 3\ \ \ ,\ \cdots\cdots \\ &\left(\frac{f_k^2}{\hbar c}\right) \sim 0.5,\ \ 0.25,\ \ 0.125,\ \ 0.083,\ \cdots\cdots \end{aligned} \qquad \left(\begin{matrix} i,\ k=1,\ 2 \\ i \neq k \end{matrix}\right). \tag{20}$$

It is of interest to see that, as numerical differences between f_1 and f_2 are not so large, contrary to the electromagnetic ones, we could not exclude either of them from the beginning. But, we should not forget the following circumstances. That is to say, Dirac's theory comes out from the simple model : at first, a charge generates an electromagnetic field around it, and there is a pole beside it, then the pole interacts with the charge through the electromagnetic field, or it arises in the reverse order. Hence, when an elementary particle have both a charge and a pole (or a mesic charge and a mesic pole), the validity of equation (15) (or (18)) is uncertain for the difficuities of the interaction of a particle with its own field, as Disac stated. Therefore, we may regard the equation (18) as the provisional formula in the following discussion.

(2). Fields and potentials which are dual to $F_{\mu\nu}$ and A_μ are defined by the formulas

$$(F^\dagger)_{\nu\lambda} = \frac{1}{2}\,\epsilon_{\nu\lambda\rho\sigma} F^{\rho\sigma}\,, \tag{21}$$

and

$$(A^\dagger)_{\nu\lambda\rho} = \epsilon_{\nu\lambda\rho\sigma} A^\sigma\,. \tag{22}$$

We can easily examine this statement in the equations (2) and (13). These formulas are very suggestive if a generalization to the meson field is intended.

In fact, we shall show later how these formulas can be applied to introduce the generalized interaction Lagrangians.

(3). If we modify the Maxwell equations to allow the magnetic pole, source functions must be added to the wave equations for the usual potentials. It can further be seen in the equation (11) or (14) that the added terms are vectors which are considered as the product of $\epsilon_{\nu\lambda\rho\sigma}$ and a trivector.

§ 3. Field equations of the spin 1 meson

We shall now attempt to verify the idea suggested above for the spin 1 meson field. In accord with the previous note (I), we shall use hereafter Kemmer's notation, namely Greek and Latin indices run from 0 to 3 and 1 to 3 respectively, and $x_0=ct$, $x_1=x$, $x_2=y$, $x_3=z$, : $g_{00}=-1$, $g_{11}=g_{22}=g_{33}=1$.

According to Kemmer, since there is no specification of the transformation of wave functions for reflexions in the spinor notation, two dual field quantities correspond to a spinor. For the spin 1 mseon field, they are

(Vector)

$$\begin{cases} \dfrac{\partial\phi_\beta}{\partial x_\alpha}-\dfrac{\partial\phi_\alpha}{\partial x_\beta}=\chi_{\alpha\beta}\,, & (23)\\[2ex] \dfrac{\partial\chi^{\alpha\beta}}{\partial x_\alpha}=\varkappa^2\phi^\beta\,, & (24) \end{cases}$$

and

(Pseudovector)

$$\begin{cases} \dfrac{\partial\phi_{\beta\gamma}}{\partial x_\alpha}-\dfrac{\partial\phi_{\alpha\gamma}}{\partial x_\beta}+\dfrac{\partial\phi_{\alpha\beta}}{\partial x_\gamma}=\varkappa^2\chi_{\alpha\beta\gamma}\,, & (25)\\[2ex] \dfrac{\partial\chi^{\alpha\beta\gamma}}{\partial x_\alpha}=\phi^{\beta\gamma}\,. & (26) \end{cases}$$

As in the formulas (21) and (22), we can postulate the following relations,

$$\begin{cases} \phi^{\alpha\beta}=\dfrac{1}{2}\,\epsilon^{\alpha\beta\gamma\delta}\chi_{\gamma\delta} \quad \rightarrow (\boldsymbol{E},\boldsymbol{H})\,, & (27)\\[2ex] \chi^{\alpha\beta\gamma}=\epsilon^{\alpha\beta\gamma\delta}\phi_\delta \quad \rightarrow (\boldsymbol{A},\phi)\,, & (28) \end{cases}$$

where $\phi_0=\phi$, $(\phi_1,\phi_2,\phi_3)=-\boldsymbol{A}$, $(\chi_{01},\chi_{02},\chi_{03})=\boldsymbol{E}$, $(\chi_{23},\chi_{31},\chi_{12})=-\boldsymbol{H}$, and $\epsilon^{0123}=1$.

Then it will be seen that equations (23) and (24) coincide with equations (26) and (25).

To show analogy with the electromagnetic field all quantities describing the meson field are denoted by the same letters, then they are

$$curl\,\boldsymbol{H}-\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}+\varkappa^2\boldsymbol{A}=0\,,\qquad div\,\boldsymbol{E}+\varkappa^2\phi=0\,, \tag{29}$$

$$\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}-grad\,\phi\,,\qquad \boldsymbol{H}=curl\,\boldsymbol{A}\,. \tag{30}$$

On the basis of the assumptions (27) and (28) we may say that the classification of the spin 1 field into two kinds of them is not necessary and the vector and pseudovector field equations are different representations of equations (29) and (30). In other words, merely the reflexion charactor is represented by the antisymmetrical tensor of fourth rank $\epsilon^{\alpha\beta\gamma\delta}$. This corresponds to the well-

known fact that Maxwell equations are only expressed by the usual field tensors $F_{\alpha\beta}$ or by the dual ones $F^{\dagger}_{\alpha\beta}$.

Now we shall study the Dirac's idea for the spin 1 -meson field by a simple example. From the equation (23) the next one is derived:

$$\frac{\partial \chi_{\alpha\beta}}{\partial x_\tau}+\frac{\partial \chi_{\tau\alpha}}{\partial x_\beta}+\frac{\partial \chi_{\beta\tau}}{\partial x_\alpha}=0\,, \tag{31}$$

or in vector notations,

$$\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}+curl\,\boldsymbol{E}=0\,, \qquad \text{and} \qquad div\,\boldsymbol{H}=0\,. \tag{32}$$

Corresponding to the currents or poles in the electromagnetic field, we are able to introduce source functions into the right-hand side of (31) or (32).

For example, if we assume the from ($c=1$)

$$\begin{cases} \dfrac{\partial \boldsymbol{H}}{\partial t}+curl\,\boldsymbol{E}=g\psi^*\alpha\psi-\dfrac{ig}{2\varkappa}\left[(grad\,\psi^*)\beta\psi-\psi^*\beta(grad\psi)\right], \\ div\,\boldsymbol{H}=-g\psi^*\psi-\dfrac{ig}{2\varkappa}\left[\dfrac{\partial\psi^*}{\partial t}\beta\psi-\psi^*\beta\dfrac{\partial\psi}{\partial t}\right], \end{cases} \tag{33}$$

then, using the Dirac wave equation for the free nucleon, Proca equations have the expressions

$$\begin{cases} \boldsymbol{E}=-\dfrac{\partial \boldsymbol{A}}{\partial t}-grad\,\phi+\dfrac{g}{2\varkappa}\psi^*\rho_3\sigma\psi\,, \\ \boldsymbol{H}=curl\,\boldsymbol{A}+\dfrac{g}{2\varkappa}\psi^*\rho_2\sigma\psi\,, \end{cases} \tag{34}$$

and

$$\begin{cases} curl\,\boldsymbol{H}-\dfrac{\partial \boldsymbol{E}}{\partial t}+\varkappa^2\boldsymbol{A}=\dfrac{g}{2\varkappa}\left[\dfrac{\partial}{\partial t}(\psi^*\rho_3\sigma\psi)+curl(\psi^*\rho_2\sigma\psi)\right], \\ div\,\boldsymbol{E}+\varkappa^2\phi=-\dfrac{g}{2\varkappa}div(\psi^*\rho_3\sigma\psi)\,, \end{cases} \tag{35}$$

or eliminating field quantities,

$$\begin{cases} \Box\boldsymbol{A}-\varkappa^2\boldsymbol{A}=-\dfrac{g}{2\varkappa}\left[\dfrac{\partial}{\partial t}(\psi^*\rho_3\sigma\psi)+curl\,(\psi^*\rho_2\sigma\psi)\right], \\ \Box\phi-\varkappa^2\phi=\dfrac{g}{2\varkappa}div\,(\psi^*\rho_3\sigma\psi)\,. \end{cases} \tag{36}$$

Equations (33), (35) and (36) correspond to Dirac's ones (4), (3) and (14) respectively. Hence, we may conclude that if the second unusual terms of the right-hand sides of (33) are disregarded, the statement (3) of § 2 holds also in the meson field. Namely, as simply verified by reverting to the tensor notation, the

source functions of (33), (34) and (35) are a trivector (vector multiplied by $\epsilon^{\alpha\beta\gamma\delta}$), a pseudotensor and a vector (trivector multiplied by $\epsilon^{\alpha\beta\gamma\delta}$).

Apart from above discussions, we shall now set up the following simple assumption as to the interactions between nucleons and spin 1 mesons.[4]

"In the usual vector (or pseudovector) meson field, besides the usual vector (or pseudovector) source function in the equation (24) (or (25)), another one may be added. The added one is a vector (or pseudovector) source function which has the form $1/6\epsilon^{\alpha\beta\gamma\delta}t_{\beta\gamma\delta}$ (or $1/6\epsilon^{\alpha\beta\gamma\delta}\tau_\delta$). The so-called tensor (or pseudotensor) source function is to be excluded."

As mentioned in § 1 tensor and pseudotensor couplings give rise to divergent results, therefore we neglect them here.

§ 4. Nuclear forces for the spin 1 meson field.

As detailed calculations have been shown in the previous note (I), we shall refer to the results briefly.

Lagrangians :

$$L_N=-\left(\frac{1}{2i}\right)\left[\psi^*(\boldsymbol{\alpha}\cdot grad\,\psi)-(grad\,\psi^*\cdot\boldsymbol{\alpha})\psi+\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi+2iM\psi^*\rho_3\psi\right], \tag{37}$$

$$L_M=-\left(\frac{1}{2}\right)\left[(curl\,\boldsymbol{A}\cdot curl\,\boldsymbol{A})-\left(\frac{\partial\boldsymbol{A}}{\partial t}+grad\,\phi\right)\cdot\left(\frac{\partial\boldsymbol{A}}{\partial t}+grad\,\phi\right)+\varkappa^2(\boldsymbol{A}\cdot\boldsymbol{A}-\phi^2)\right], \tag{38}$$

$$L_1=g_1v_\alpha\phi^\alpha+\frac{1}{6}g_1't_{\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma\delta}\phi_\delta \tag{39}$$

$$=-\psi^*[g_1(\boldsymbol{\alpha}\cdot\boldsymbol{A}-\phi)+g_1'(\boldsymbol{\sigma}\cdot\boldsymbol{A}-\rho_1\phi)]\psi\,, \tag{40}$$

where $$(v_0, v_i)=(-\psi^*\psi,\ \psi^*\boldsymbol{\alpha}\psi)$$

and $$(t_{0ik},\ t_{ikl})=(\psi^*\boldsymbol{\sigma}\psi_l,\ -\psi^*\gamma^5\psi)\,. \qquad (\hbar=c=1)\,. \tag{41}$$

Hamiltonians :

$$H=H_N+H_M+H_1'+H_{int} \tag{42}$$

$$H_N=\left(\frac{1}{2i}\right)\int[\psi^*(\boldsymbol{\alpha}\cdot grad\,\psi)-(grad\,\psi^*\cdot\boldsymbol{\alpha})\psi+2iM\psi^*\rho_3\psi]\,d\tau, \tag{43}$$

$$H_M=\left(\frac{1}{2}\right)\int\left[(curl\,\boldsymbol{A}\cdot curl\,\boldsymbol{A})+\varkappa^2(\boldsymbol{A}\cdot\boldsymbol{A})+\left(\frac{1}{\varkappa^2}\right)(div\,\Pi_1)^2+(\Pi_1\cdot\Pi_1)\right]d\tau, \tag{44}$$

$$H_1'=\int \psi^*\left[g_1\left(\boldsymbol{\alpha}\cdot\boldsymbol{A}-\frac{1}{\varkappa^2}\,div\,\Pi_1\right)+g_1'\left(\boldsymbol{\sigma}\cdot\boldsymbol{A}-\frac{1}{\varkappa^2}\,\rho_1\,div\,\Pi_1\right)\right]\psi dV, \tag{45}$$

$$H_{di}=\frac{1}{2}\left[\left(\frac{g_1}{\varkappa^2}\right)^2(\psi^*\psi)^2+\left(\frac{g_1'}{\varkappa^2}\right)^2(\psi^*\rho_1\psi)^2\right.$$

$$\left.+\left(\frac{g_1 g_1'}{\varkappa^2}\right)\{(\psi^*\rho_1\psi)(\psi^*\psi)+(\psi^*\psi)(\psi^*\rho_1\psi)\}\right]dV, \tag{46}$$

where

$$\Pi_1=\frac{\delta L}{\delta\left(\frac{\partial \boldsymbol{A}}{\partial t}\right)}=\left(\frac{\partial \boldsymbol{A}}{\partial t}+grad\,\phi\right). \tag{47}$$

The nuclear potential part in the Hamiltonian which is obtained from (42) by a canonical transformation[6] takes the following form:

$$H=H_{v.}+H_{p.v.}+H_{m.}, \tag{48}$$

$$H_{v.}=\frac{g_1^2}{(E_{n_1}-E_{m_1})^2-k_0^2}\left[(\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)})-1\right], \tag{49}$$

$$H_{p.v.}=\frac{g_1'^2}{(E_{n_1}-E_{m_1})^2-k_0^2}\left[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})-\rho_1^{(1)}\rho_1^{(2)}+\frac{4M^2}{\varkappa^2}\,\rho_2^{(1)}\rho_2^{(2)}\right.$$

$$\left.+\frac{2iM}{\varkappa^2}(\rho_2^{(1)}\rho_1^{(2)}+\rho_2^{(2)}\rho_1^{(1)})(E_{n_1}-E_{m_1})\right], \tag{50}$$

$$H_{m.}=\frac{g_1 g_1'}{(E_{n_1}-E_{m_1})^2-k_0^2}\left[(\boldsymbol{\alpha}^{(2)}\cdot\boldsymbol{\sigma}^{(1)})+(\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})-(\rho_1^{(1)}+\rho_1^{(2)})\right.$$

$$\left.+\frac{2iM}{\varkappa^2}(\rho_2^{(1)}+\rho_2^{(2)})(E_{n_1}-E_{m_1})\right]. \tag{51}$$

It is to be noted that the direct (or contact) interaction terms vanish entirely in this result.* This means that in the Tomonaga-Schwinger formalism the normal dependent terms of Hamiltonians cancel out in the second order nuclear potentials.

Then, the expression (48) is transformed into the form which contains only $\boldsymbol{\sigma}$ operator by the same method as adopted in § 4 of (I)[3]. Thus we find the nuclear potential

$$W_1=-\left[-g_1^2+g_1'^2(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})-g_1'^2\left\{\frac{1}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})+\omega(r)S_{12}\right\}\right.$$

---

* In (I) the signs of last terms of (51) and (56) were misprinted. The correct forms are $-H_{p.v.div}$ and $-H_{m.div}$.

$$+\left(\frac{1}{2M}\right)g_1 g_1'\{(\sigma^{(1)}\cdot\sigma^{(2)})-1\}\{(\sigma^{(1)}\cdot\boldsymbol{P}_1)+(\sigma^{(2)}\cdot\boldsymbol{P}_2)\}$$

$$+\left(\frac{\varkappa}{2M}\right)^2(g_1^2-g_1'^2)\left\{\frac{1}{3}(\sigma^{(1)}\cdot\sigma^{(2)})+\omega(r)S_{12}\right\}\{(\sigma^{(1)}\cdot\sigma^{(2)})+1\}\Big]\left(\frac{1}{4\pi}\right)\frac{e^{-\varkappa r}}{r}\,, \tag{52}$$

where

$$\omega(r)=\left(\frac{1}{3}+\frac{1}{\varkappa r}+\frac{1}{\varkappa^2 r^2}\right)\quad\text{and}\quad S_{12}=\left(\frac{3(\sigma^{(1)}\cdot\boldsymbol{r})(\sigma^{(2)}\cdot\boldsymbol{r})}{r^2}-(\sigma^{(1)}\cdot\sigma^{(2)})\right). \tag{53}$$

or in the limit of $M\to\infty$

$$W_1=-T\Big[-g_1^2+g_1'^2(\sigma^{(1)}\cdot\sigma^{(2)})-g_1'^2\left\{\frac{1}{3}(\sigma^{(1)}\cdot\sigma^{(2)})+\omega(r)S_{12}\right\}\Big]\left(\frac{1}{4\pi}\right)\frac{e^{-\varkappa r}}{r}\,, \tag{54}$$

where T is added to consider the symmetrical and charged theories,

$$T=(\tau^{(1)}\cdot\tau^{(2)})/2\qquad\text{or}\qquad T=(\tau_1^{(1)}\tau_1^{(2)}+\tau_2^{(1)}\tau_2^{(2)})/2\,. \tag{55}$$

This potential is nothing but the ordinary one which is obtained by mixing vector meson with the vector coupling and the pseudovector meson with the pseudovector coupling. It is, however, remarked that the apperance of $1/r^3$ singularity in (54) is due to the inadequate approximation and by considering recoil effects of nucleons this singularity is removable, while the $1/r^3$ singularity of the potential by the pseudovector meson by the old perturbation method came out from the direct interaction term, therefore it is impossible to avoid it by taking account of the recoil effect. Detailed account will be given in a future paper.

§ 5. Discussions

One of the interests in this note is to see whether the spin 1 field is preferable to explain the nuclear phenomena in place of the conventional pseudoscalar meson theory or not. From the above calculation it is impossible to derive the decisive conclusion at the present stage, but we shall point out a few important results.

(1) As regards the coupling constants, it is the first problem to infer which is larger, $\left(\frac{g_1^2}{4\pi\hbar c}\right)$ or $\left(\frac{g_1'^2}{4\pi\hbar c}\right)$. Considering the anomalous magnetic moments of nucleons which have been evaluated by many authors, the estimated order of magnitude of them are ~ 4 in the vector theory, on the other hand, the pseudovector theory gives the reasonable ones. Hence, it seems to be natural that we assume

$$|g_1|>|g_1'|\,.$$

When the numerical difference between them is large, we may give up the vector coupling in the spin 1 field and retain only the pseudovector one for the same reason as magnetic poles are excluded in the Maxwell-Dirac theory. As to magnetic moments themselves, it is necessary further investigations to obtain the correct results because of the ambiguity of meson currents.[7]

(2) It is to be noted that, if we select the models which give rise to the correct signs of the quadrupole moment of the deuteron (i.e. positive sign) out of Kemmer's four types of mesons with the exception of tensor couplings and the neutral theory, we get the next ones

$$\text{Charged and Symmetrical}\begin{Bmatrix}\text{Pseudoscalar}\\ \text{Pseudovector}\end{Bmatrix}\text{meson fields.}$$

In fact in the latter case, it is easily confirmed in the expression (54).

Finally, we are able to say that the spin 1 field with or without *vector coupling* is as remarkable as the pseudoscalar meson field.

The author wishes to express his thanks to Prof. H. Yukawa and Assist. Prof. T. Inoue for their kind interests to this work.

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Errata (Vol. V, No. 1)

The arguments of § 4 and § 5 of (I) are incorrect. That is, the cross terms are dropped there. Terms of the form

$$\frac{1}{E}\{2i\phi f_3' f_1(\boldsymbol{\sigma}_1\cdot\boldsymbol{P}_1+\boldsymbol{\sigma}_2\cdot\boldsymbol{P}_2)-2i f_3' f_1(\boldsymbol{\sigma}_1\cdot\boldsymbol{P}_1+\boldsymbol{\sigma}_2\cdot\boldsymbol{P}_2)\phi\}$$

in two brackets of Eq. (75), and

$$\mp\frac{i}{M}f_3' f_1(\boldsymbol{\sigma}_1\cdot\boldsymbol{P}_1+\boldsymbol{\sigma}_2\cdot\boldsymbol{P}_2)\phi$$

in the brackets of Eqs. (76) and (78) should be added respectively.

However, in the limit of $M\to\infty$ these vanish.

Letters to the Editor

Notes on Fluctuation in Nuclear Evaporation Process

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In this previous paper[1], we have given a very brief discussion of fluctuation problem connected with nuclear evaporation processes. Thus in this note we shall give some complementary considerations about this problem.

For brevity, we assume that each particle (j) evaporated from excited nucleus (with temperature T and excitation energy X, $T=T(X)$) takes away a definite energy

$$E^{(j)}=E_b+V+f(X), \quad f(X)\sim 2T, \qquad (1)$$

where E_b=binding energy, V=Coulomb barrier height } for this particle.

We will not treat here the fluctuation of the number of evaporated particles arising from the energy fluctuation of these particles (cf. I). Furthermore, we do not take account of the change of binding energies, and of mass or atomic number of nucleus. However, the possibility that the nucleus can emit various kinds of particles results in the fluctuation of the number of ejected particles of each kind. Let us consider this fluctuation, using the method given by Horning and Baumhoff[2].

For simplicity, we treat the simplest case where the excited nucleus evaporates only protons and neutrons. It is easy to extend our treatment to more general cases, where the other particles may be emitted.

Let $p(\pi, \nu; X)dX$ be the probability that just π protons and ν neutrons are emitted from the nucleus, when nuclear excitation falls into a interval $(X, X+dX)$ from its initial value X_0 in the course of evaporation. Then, if we put

$$\frac{1}{\varepsilon(X)}=\frac{\Gamma_n E^{(n)}+\Gamma_p E^{(p)}}{\Gamma_n+\Gamma_p},$$

$$E^{(n)}=E_b+f(X),$$

$$E^{(p)}=E_b+V+f(X),$$

and

$$S(X)=\int_0^X \varepsilon(X)dX, \quad S_0=S(X_0),$$

we find the following differential equation

$$\frac{\partial p(\pi, \nu; S)}{\partial S}=p(\pi, \nu; S)-\frac{\Gamma_n}{\Gamma_n+\Gamma_p}p(\pi, \nu-1; S)-\frac{\Gamma_p}{\Gamma_n+\Gamma_p}p(\pi-1, \nu; S). \qquad (2)$$

In these equations Γ_n and Γ_p mean the neutron- and proton-width, respectively. If the ratio Γ_n/Γ_p does not depend on X (otherwise one may take suitable average value of Γ_n/Γ_p), the solution of (2) under the initial condition

$$p(\pi, \nu; S_0)=\delta_{\pi 0}\delta_{\nu 0}$$

is, as is well known, given by the double Poisson distribution:

$$p(\pi, \nu; S)=\frac{1}{\pi!\,\nu!}\left(\frac{S_0-S}{\Gamma_n+\Gamma_p}\right)^{\pi+\nu}\Gamma_p^{\pi}\Gamma_n^{\nu}e^{-(S_0-S)} \qquad (2')$$

If we do not interest in the number of ejected neutrons, we have only to use the margin distribution

$$\sum_{\nu=0}^{\infty} p(\pi, \nu; S)=p(\pi, S).$$

From (2) we obtain

$$\frac{\partial}{\partial Q}p(\pi, Q)=p(\pi, Q)-p(\pi-1, Q), \qquad (2)$$

where we introduce the new variable:

$$Q=\frac{\Gamma_p}{\Gamma_n+\Gamma_p}S.$$

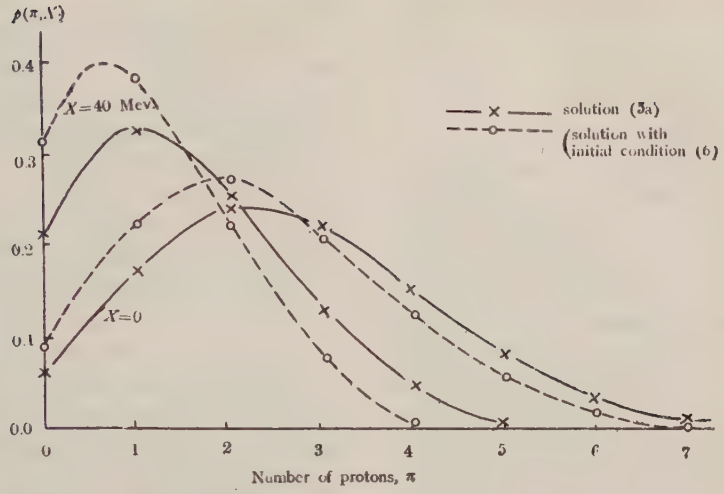

Fig. 1 Comparison of the solution (5a) with the solution obtained by Horning-Baumhoff's method, for the initial excitation: $X_0=100$ Mev. and the mass number $A=100$. (We assume that $E^{(p)}=14+2T$, $E^{(n)}=8+2T$, $\Gamma_p=1$ and $\Gamma_n=1.5$ in Mev.)

If the ratio Γ_p/Γ_n does not depend on X, one finds

$$Q=Q(X)=\int_0^X q(X)dX,\ \frac{1}{q(X)}=E^{(p)}+\frac{\Gamma_n}{\Gamma_p}E^{(n)}.$$

Similarly, for the margin distribution

$$p(\nu, S)=\sum_{\pi=0}^{\infty}p(\pi, \nu, S)$$

we get

$$\frac{\partial}{\partial R}p(\nu, R)=p(\nu, R)-p(\nu-1, R),$$

$$R=\frac{\Gamma_n}{\Gamma_p+\Gamma_n}. \qquad (4)$$

Evidently, the Poisson distributions:

$$p(\pi, Q)=e^{-(Q_0-Q)}\cdot\frac{(Q_0-Q)^{\pi}}{\pi!},\quad Q_0=Q(X_0), \qquad (5a)$$

$$p(\nu, Q)=e^{-(R_0-R)}\cdot\frac{(R_0-R)^{\nu}}{\nu!},\quad R_0=R(X_0), \qquad (5b)$$

satisfy the equations (2) and (4). Thus, as for margin distributions, the expectation in I is confirmed. Furthermore, as is readily seen from (2), the distribution for $\pi+\nu$ is also Poisson type. This distribution shows large fluctuation, though the fluctuation caused by the energy spectrum of individual particle is more important (cf. I).

Now, $p(\pi, Q)$ corresponds to $p_\pi(Q)$ used by Horning and Baumhoff.[2] The solutions (5), however, belongs to different initial conditions than the ones adopted by them, i.e.

$$p(\pi, Q)=0 \quad \text{for} \quad Q>Q_\pi=Q(X_0-\pi T_p),$$
$$T_n=E_b+V. \qquad (6)$$

The mean value $\bar{\pi}(Q)$ or $\bar{\nu}(R)$ in our case is larger than the one obtained by Horning and Baumhoff. This situation is explained by Fig. 1.

Finally, we discuss the relations to the "mean cooling law". If we approximate the equation (3) by the differential equation

$$\frac{\partial p}{\partial Q}=\frac{\partial p}{\partial \pi}, \qquad (3')$$

with the initial condition

$$p(\pi, Q_0)=\delta(\pi),$$

we get

$$p(\pi, Q)=\delta(Q+\pi-Q_0)$$

or

$$\pi_{\text{m.c.l.}}(Q)=Q_0-Q.$$

This is just equal to the mean value obtained from (5):

$$\bar{\pi}(Q)\equiv\sum_{\pi=0}^{\infty}\pi p(\pi,\ Q)=Q_0-Q.$$

Entirely the same result is obtained for neutrons. In general, similar results hold for any kinds of evaporated particles. Thus, we believe that the contents of mean cooling law are considerably clarified. Namely, we may conclude as follows: The number of evaporated particles from the mean cooling law of differential type is the mean value. Actual number obeys Poisson distribution. Of course, these are valid only for our simplified case described above. It may, however, hoped that the main features of fluctuation would be roughly preserved in actual cases.

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On the Stars of Light Nuclei

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Recently, Fermi[1] has discussed the multiple meson production by a purely statistical method. He derived the distribution of number of produced mesons, assuming that possible final states occur with frequencies proportional to their statistical weights. Such a method could be used to examine the disintegration modes of excited nuclei. In fact, as for heavy nuclei the statistical theory was successfully applied to various reactions[2]. However, the "stars" of light nuclei were not treated so far along this line. It is the purpose of this note to discuss this problem.

Firstly, let us consider the negative pion capture process in a C^{12}-nucleus. (The pion stars of heavy nuclei were discussed formerly[3].) If a pion is captured, its rest energy is dumped in the volume Ω, which can be supposed to be of the order of nuclear volume $(4\pi/3)R^3$, R=nuclear radius, and contains seven neutrons and five protons in our case. More precisely, we may regard Ω as an adjustable parameter. As was stated before, the probability of transition into the final state is assumed to be proportional to the volume S of phase space incorporated with the total energy $T=\mu c^2-B$, where $\mu c^2\fallingdotseq 140$ Mev is rest energy of a pion and B means the decrease of available energy due to binding energies. (We use Fermi's notations, cf. ref. 1).) T is roughly determined by the number ν_α of alpha-particles existing in the final state:

$T\sim 60$, 80 and 100 Mev
for $\nu_\alpha=0$, 1 and 2, respectively.

For given ν_α, S is larger for increasing the number ν of particles in a final state. While for the smaller ν_α the larger becomes S, since the increase of T overcomes the decrease of ν. Therefore, in our case the final state 2 alpha-particles+3 neutrons+1 proton, i.e., the three prong-star, would have largest probability and probabilities of other states are smaller than this.

Similar results are obtained for pion stars due to N or O nuclei. Thus we may conclude that the size distribution of pion stars of light nuclei shows a considerably sharp maximum at 3~4-prongs. This result is in agreement with the fact known from the recent photographic experiments[4].

Next we shall examine the reactions of light nuclei bombarded with high energy particles ($\lesssim$150 Mev). As can be seen from the case of pion stars, even when $T\sim 150$ Mev the most probable final state would contain rather few prongs. If T is small, the transitions to final states with larger ν are completely forbidden or allowed with only very small probabilities. And the most probable number of prongs will not change drastically with T. Furthermore, on account

of the nuclear transparency the available energy T of nuclear reaction shows very broad distribution[5]. Thus we may expect that the prong distribution would be almost insensitive to the incident energy of bombarding particles. This fact was confirmed by Berkeley's experiments[6].

These considerations may be regarded as an evidence for what would be expected qualitatively from the so-called alpha-particle model for a nucleus.

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On the Effect of γ-Ray to the Cosmic-Ray Underground

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Recently, the production of π-meson by high energy γ-rays ($\sim$300 Mev) is precisely analysed.[1] We have studied the effect of the high energy γ-rays in the cosmic-ray phenomena. Our purpose is to obtain phenomenologically the character of the γ-π process in high energy region ($\sim 10^9$ev or more). Another aim is to make clear up the problem of the model in the cosmic-ray.

In many experiments,[2] it has been pointed out that γ-rays in cosmic-ray produce the penetrating particle. Tabin[3] has obtained the nuclear geometrical cross section as the production cross section of the penetrating particle by γ-rays. Considering that the cross section of the ordinary pair creation is comparable with the geometrical cross section of air nucleus, the value obtained by Tabin seems not so decisive—that is, if it were so, the ratio of the hard and soft-component in air would change appreciably. As another effect of γ-rays to the cosmic-ray phenomena, one of the authors has pointed out the possibility of its effect to the positive excess of the hard component.[1] It is due to the asymmetry of the production of negative and positive π-meson by γ-rays.

Now, μ-meson and the soft component in equilibrium with μ-meson are the principal parts of the cosmic-ray underground. Nuclear component is of extremely small fraction. Evans and George[5] have obtained the following conclusion from the frequency of the star production in their experiment at 60m H_2O depth underground. That is, μ-meson's cross section for star production must be of the magnitude of 0.4×11^{-28} cm^2 per nucleon. The result is inconsistent with two-meson theory. Another results against two-meson model is that obtained by Forró et al.[6] at the depth of 1000 mH_2O underground. According to them the intensity of the cosmic-ray measured by three coincidence decreases when 10cm thickness of lead is set between two counters, while it gradually increases with the thickness of lead, indicating the maximum near 50cm. Rapid fall occurs only when 10cm Pb is put between upper two counters.

In order to explain these results by the contribution of π-meson produced by γ-π process, we estimate the fraction of π-meson relative to μ-meson intensity at each depth underground. At first, we must inform about the interaction of π-meson with matter, especially the collision mean free path. The information, however, is not so precise, though many experiments are attempted.[7] In spite of the lack of the knowledge, the interaction of π-meson with matter may be thought to be quite strong, as is seen from the con-

sideration correlating to the nucleon-nucleon (or nucleus) interaction. Accordingly assuming the nuclear cascade process of π meson, we have calculated the transition curve of π-meson in matter based on the following assumption: i) π-meson with the kinetic energy more than its own rest energy acts cascade -likely.[8] ii) π-meson loses its energy by a half, passing through one collision mean free path (l) and creates another π-meson. We remain l undetermined until the end of the numerical calculation. Thus, the obtained fraction of the intensity of π-meson whose kinetic energy is larger than the rest energy ($\varkappa$) in equilibrium with μ-meson is presented in Table 1.

Table 1. Fraction of π-meson relative to μ-meson

$I_\pi(>2\varkappa)/I_\mu=(\phi_0 \cdot N/A \cdot l)\times$	3.4	6.6	16×10^{-2}
depth (in mH_2O)	60	200	1000

ϕ_0 is the mean cross section of γ-π process per nucleus. N and A are the number and of nucleon per gram and the atomic weight of the matter underground, respectively. We take $A=16$.

a) If the star agent at $60mH_2O$ depth is thus produced π-meson, we can estimate the value of ϕ_0 independently on l.

$$\phi_0=4.8\times10^{-26}\,\mathrm{cm^2/nucleus}. \qquad (1)$$

We note that the mean energy of γ-rays at this depth is about 10 Bev. Of course, the π-meson production by γ-ray will be accompanied with the nucleon recoil, which may cause nuclear evaporation. The value (1) is also required when the star agent is γ-ray itself. The further experiment will determine whether the agent is ionizing or nonionizing. We may expect comparable contribution due to these two effects.

b) Taking the value (1), we obtain $I_\pi/I_\mu=0.21\times10^{-3}l$ at the depth of 1000 mH_2O. The π-meson in lead multiplicates and indicates the maximum in intensity at the distance l or more apart. If we take the geometrical cross section of the nucleus as the π-nucleus interaction, $l\simeq100\mathrm{g/cm^2}$. This value of l gives $I_\pi/I_\mu\simeq2\%$ which is slightly small to explain the Forró's results. We remark, however, that the π-meson's contribution may increase, if the multiple production of π-meson in π-nucleon interaction and π^0-meson in the above process are taken into account. Considering that π-meson at 1000 mH_2O is accompanied with shower (another π-meson or due to π^0-meson), we may be able to explain the decrease of the intensity caused by the insertion of 10 cm Pb between upper counters. Because smaller energy part may be absorbed. We also remark that the value (1) is quite reasonable to explain the fact that the soft component in air shower is accompanied with the penetrating particle by a few percent. Detailed paper will be reported near future.

In conclusion, the authors should like to express their cordial thanks to Prof. S. Sakata for his kind interest in their work.

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The Lateral and Angular Distribution of Cascade Showers

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The lateral and angular spread of cascade showers are of great importance for the interpretation of air showers, and hitherto have been treated by various authors. The structure function obtained by Molière[1] is the most available one at this time, but it was pointed out[2] that this function does not represent the structure of showers accurately.

In order to obtain the exact structure function at any depth, we computed it analytically under the assumption "Approximation B"[3], since the inaccuracy of the Molière function is mainly due to his numerical calculation based on the Arley's approximation.

We started from the Landau's[4] equation and solved it analytically using the Mellin, Laplace and Fourier transformations. The course of our calculation, however, is somewhat complicated and we shall give only the final results† in this paper.

The differential lateral structure function $\pi_1(E_0, E, r, t)2\pi r dr$, which is the average number of electrons having the energy E and the deviation from the axis between r and $r+dr$ at the thickness t, is given by

$$\pi_1 = -\frac{1}{8\pi^4 i}\iiint_C ds dp dq\, I'(1+p) I'(-q) \times \mathfrak{M}_1(p, q, t)\left(\frac{E_0}{E}\right)^s \frac{1}{E}\left(\frac{E}{K}\right)^2\left(\frac{E^2 r^2}{K^2}\right)^{-p-1}\left(\frac{\epsilon}{E}\right)^q, \tag{1}$$

where s, p, q are the complex variables, and the integration paths C are running parallel to the imaginary axis. $\mathfrak{M}_1(p, q, t)$ is defined by, in the notation of Rossi-Greisen[3],

$$\left\{\frac{\partial^2}{\partial t^2} + (A(s+2p+q)+\sigma_0)\frac{\partial}{\partial t} + (A(s+2p+q)\sigma_0 - B(s+2p+q)C(s+2p+q))\right\}\mathfrak{M}_1(p, q, t) = \left(\frac{\partial}{\partial t}+\sigma_0\right)\{p(\xi-t)^2\mathfrak{M}_1(p-1, q, t) + (s+2p+q)q\mathfrak{M}_1(p, q-1, t)\} \tag{2}$$

with

$$\mathfrak{M}_1(0, 0, t) = H_1(s)e^{\lambda_1(s)t} + H_2 e^{\lambda_2(s)t}. \tag{3}$$

Since this equation is similar to that obtained by Snyder[5] and Bhabha-Chakrabarty[6], it can be solved in the same way.*

The total lateral structure function $\Pi_1(E_0, 0, r, t)$ is given by

$$\Pi_1(E_0, 0, r, t) = \int_0^\infty \pi_1 dE. \tag{4}$$

Integration with respect to E and q yields

$$\Pi_1(E_0, 0, r, t) = -\frac{1}{4\pi^3}\iint_C ds dp\, I'(1+p) \times I'(s+2p)\mathfrak{M}_1(p, -2p-s, t) \times \left(\frac{E_0}{\epsilon}\right)^2\left(\frac{\epsilon}{K}\right)^2\left(\frac{\epsilon^2 r^2}{K^2}\right)^{-p-1}. \tag{5}$$

Similarly the differential and total angular structure functions are given by

$$\pi_2(E_0, E, \theta, t) = -\frac{1}{8\pi^4 i}\iiint_C ds dp dq\, I'(1+p) \times I'(-q)\mathfrak{M}_2(p, q, t) \times \left(\frac{E_0}{E}\right)^s \frac{1}{E}\left(\frac{E}{K}\right)^2\left(\frac{E^2\theta^2}{K^2}\right)^{-p-1}\left(\frac{\epsilon}{E}\right)^q, \tag{6}$$

$$\Pi_2(E_0, 0, \theta, t) = -\frac{1}{4\pi^3}\iint_C ds dp\, I'(1+p) I'(2p+s) \times \mathfrak{M}_2(p, -2p-s, t)\left(\frac{E_0}{\epsilon}\right)^s\left(\frac{\epsilon}{K}\right)^2\left(\frac{\epsilon^2\theta^2}{K^2}\right)^{-p-1}, \tag{7}$$

where $\mathfrak{M}_2(p, q, t)$ is defined by

$$\left\{\frac{\partial^2}{\partial t^2} + (A(s+2p+q)+\sigma_0)\frac{\partial}{\partial t} + (A(s+2p+q)\sigma_0 - B(s+2p+q)C(s+2p+q))\right\}\mathfrak{M}_2(p, q, t) = \left(\frac{\partial}{\partial t}+\sigma_0\right)\{p\mathfrak{M}_2(p-1, q, t) + (s+2p+q)q\mathfrak{M}_2(p, q-1, t)\} \tag{8}$$

† For the check of our calculation, we computed the total number of electrons from these distribution functions, which gave the same results obtained by Snyder,[5] as it must be.

* $\mathfrak{M}_1(p, q, t)$ is given by putting $\xi - t$ (variable of the solution of eq. (2)) $\to 0$.

with

$$\mathfrak{M}_2(0, 0, t)=\mathfrak{M}_1(0, 0, t). \tag{9}$$

For the comparison with the results obtained by others, we first computed the mean square deviation and angle of deflection defined by

$$\langle r^2\rangle_{AV}=\frac{\int_0^\infty r^2\pi_1 2\pi r dr}{\int_0^\infty \pi_1 2\pi r dr} \tag{10}$$

and

$$\langle\theta^2\rangle_{AV}=\frac{\int_0^\infty \theta^2\pi_2 2\pi\theta d\theta}{\int_0^\infty \pi_2 2\pi\theta d\theta} \tag{11}$$

Making use of our distribution functions (eqs. (1), (6)) and limiting ourselves to the shower maximum and $E\to\infty$, we can obtain the following results:

$$\langle r^2\rangle_{AV}=\frac{\mathfrak{M}_1(1, 0, t)}{\mathfrak{M}_1(0, 0, t)}\left(\frac{K}{E}\right)^2=0.723\left(\frac{K}{E}\right)^2, \tag{10}'$$

$$\langle\theta^2\rangle_{AV}=\frac{\mathfrak{M}_2(1, 0, t)}{\mathfrak{M}_2(0, 0, t)}\left(\frac{K}{E}\right)^2=0.569\left(\frac{K}{E}\right)^2. \tag{11}'$$

The results of various authors are listed in Table I.

Table I. Comparison of shower spread at high energies obtained by various authors.

	$\langle r^2\rangle_{AV}\left(\frac{E}{K}\right)^2$ (radi. unit)	$\langle\theta^2\rangle_{AV}\left(\frac{E}{K}\right)^2$ (radian)
Euler-Wergeland[7]	0.074	0.33
Moliere	0.835	0.6
Belenky[8]	0.94	—
Janossy*[9]	0.724	0.570
Roberg-Nordheim[10]	0.642	0.545
Ours	0.723	0.569

* These values are derived using the method of Roberg-Nordheim.[10]

It must be noted that our values are very close to that obtained by Janossy which are the most accurate ones so far.

The most remarkable results derived from equation (5) is the behaviour of the total structure function near the core. Using the pole of $\Gamma(1+p)\Gamma(2p+s)$, we can get the following results.

$$\Pi_1\sim r^{s-2} \qquad \text{for } s<2, \tag{12}$$

$$\Pi_1\sim\text{const.} \qquad \text{for } s>2. \tag{12}'$$

These results are similar to that obtained by Pomeranchuk[11] and Migdal[12] but the definition of s is different from theirs, provided $E_0\neq\infty$. In our case, s is defined by

$$\lambda_1'(\ )t+ln\left(\frac{E_0}{\epsilon}\right)+ln\left(\frac{\epsilon r}{K}\right)=0, \tag{13}$$

while their s is defined by

$$\lambda_1'(s)t+ln\left(\frac{E_0}{\epsilon}\right)=0. \tag{14}$$

As $\lambda_1'(s)\sim-\frac{1}{s^2}$, eqs. (13) (14) become

$$s^2=t/\left\{ln\left(\frac{E_0}{\epsilon}\right)+ln\left(\frac{\epsilon r}{K}\right)\right\} \tag{13}'$$

$$s^2=t/ln\left(\frac{E_0}{\epsilon}\right). \tag{14}'$$

As is easily seen from these equations, s is slightly larger than theirs, and their results must be interpreted as the accurate ones only for $E_0=\infty$. Our structure function near the core is less steep compared with theirs. The physical significance of this result is that the core is composed of the high energy particles having the spectrum deviated from the single power law when E is finite.

Detailed accounts and some applications to cosmic ray phenomena will be given in the subsequent paper.

The authors wish to express their appreciation to Dr. Blatt, Messrs. Hayakawa and Fujimoto for their valuable advice throughout this work.

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On the Equivalence of Pseudoscalar and Pseudovector Coupling Constants in Pseudoscalar Meson Theory

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The problem of the equivalence of pseudoscalar and pseudovector couplings in pseudoscalar meson theory has been examined by several authors and especially Case[1] proved that to the second order in the coupling constants the two couplings are equivalent for the calculation of nuclear forces and for many other simpler problems using the Schwinger-Tomonaga's many-time formalism by carrying through the successive two canonical transformations upon the Schrödinger equation in order to eliminate the divergence terms. In this short note we shall give a very simple proof that the two couplings are equivalent for the second order approximation. This proof consists in eliminating odd operators[2] from Dirac equation for a nucleon in order to obtain Pauli type equation. This can be done by a canonical transformation on the Dirac Hamiltonian for a nucleon interacting with external meson field. The Schrödinger equation in the case of charged pseudoscalar meson field is given by

$$i\hbar\partial\Psi/\partial t=H\Psi, \tag{1}$$

$$H=c\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{p}+\rho_3Mc^2+f(\tau_+\rho_2U^*+\tau_-\rho_2U)$$
$$+\frac{g}{\varkappa}\Big(\tau_+\boldsymbol{\sigma}\cdot\mathrm{grad}U^*+\tau_-\boldsymbol{\sigma}\cdot\mathrm{grad}U$$
$$+\tau_+\rho_1\frac{1}{c}\frac{\partial U^*}{\partial t}+\tau_-\rho_1\frac{1}{c}\frac{\partial U}{\partial t}\Big), \tag{2}$$

with

$$\tau_+=(\tau_1+i\tau_2)/2,\quad \tau_-=(\tau_1-i\tau_2)/2,$$
$$\varkappa=\mu c/\hbar$$

where U is the meson field quantity, M the nucleon mass, μ the meson mass, f the pseudoscalar coupling constant, g the pseudovector one and other notations are usual ones. If S is an Hermitian operator, then the transformation,

$$\Psi'=e^{iS}\Psi, \tag{3}$$

$$H'=e^{iS}He^{-iS}-i\hbar e^{iS}(\partial e^{-iS}/\partial t), \tag{4}$$

leaves Eq. (1) in the Hamiltonian form:

$$i\hbar\partial\Psi'/\partial t=H'\Psi'. \tag{5}$$

If we write the Hamiltonian in the form

$$H=\beta Mc^2+E+O, \tag{6}$$

where E is an even operator and O is an add operator, the canonical transformation generated by the Hermitian operator[3]

$$S=-(i/2Mc^2)\beta O \tag{7}$$

gives the new Hamiltonian[4]

$$H'=\beta Mc^2+E+\frac{\beta}{2Mc^2}O^2$$
$$-\frac{1}{8M^2c^4}\Big[O,[O,E]+i\hbar\frac{\partial O}{\partial t}\Big]$$
$$+\frac{i\hbar}{2Mc^2}\beta\frac{\partial O}{\partial t}+\frac{\beta}{2Mc^2}[O,E]$$
$$-\frac{1}{3M^2c^4}O^3+\cdots\cdots, \tag{8}$$

which contains no odd terms of order $(1/Mc^2)^0$. Thus we can successively remove odd terms from the Hamiltonian to any desired order in $(1/Mc^2)$. When the kinetic energy of a nucleon and the interaction energy with meson field are small compared with Mc^2, above expansion may be hoped to converge sufficiently rapidly. It is very easy to apply this general procedure to the Hamiltonian (2) and if we remove odd terms of $(1/Mc^2)^0$, $(1/Mc^2)^1$ and $(1/Mc^2)^2$ from the Hamiltonian, the results are

$$H_f' = \beta Mc^2 + \frac{\beta}{2M}p^2 + \frac{\hbar}{2Mc} f \sigma \cdot \mathrm{grad}(\tau_+ U^* + \tau_- U) + \frac{\beta}{2Mc^2} f^2 (\tau_+ U^* + \tau_- U)^2, \quad (9)$$

and

$$H_g' = \beta Mc^2 + \frac{\beta}{2M}p^2 + \frac{g}{\chi} \cdot \sigma \mathrm{grad}(\tau_+ U^* + \tau_- U) - \frac{1}{8M^2c^3}[c\rho_1 \sigma \cdot p[c\rho_1 \sigma \cdot p, \frac{g}{\chi} \sigma \cdot \mathrm{grad}(\tau_+' U^* + \tau_- U)]], \quad (10)$$

for pure pseudoscalar and pseudovector couplings respectively, where we employed static approximation (putting time derivatives to zero). The last term in (10) is small in non-relativistic case, the last term in (9) the higher order one. Considering that U itself contains coupling constants implicitly, we can see readily the two couplings are equivalent to f^2 or g^2 in non-relativistic case. If the following relation is valid;

$$(\mu/2M)f = g, \quad (11)$$

the contributions to static non-relativistic nuclear forces are entirely the same for two couplings and the nuclear potential is for both couplings given by

$$-\left(\frac{g}{\chi}\right)^2 \frac{\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)}}{2} (\sigma^{(1)} \cdot \mathrm{grad}^{(1)})(\sigma^{(2)} \cdot \mathrm{grad}^{(2)}) \frac{e^{-\chi|r_1 - r_2|}}{|r_1 - r_2|}, \quad (12)$$

which can be obtained most easily by the method of Green's function. Eq. (11) was derived by Araki[5] by calculating directly the non-relativistic matrix element of $\sigma \cdot \mathrm{grad}\, U$. If we leave the time derivatives of field quantities, the equivalence condition is changed to the form $g = (\mu/4M)f$. In other types of meson fields the above method does not give any definite result.

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On the *D*-Function of Non-local Fields

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September 5, 1950

Recently Yukawa[1] developed a non-local field theory with the aim of eliminating difficulties of ordinary local fields. Although the effective differences between local and non-local theories will appear only when the interaction between fields is introduced[2], it may be interesting to study its physical meaning within the limits of free field. As is well known, the ambiguities of ordinary theories arise from the singularity of *D*-function on the light cone, we tried therefore first to examine the form and behavior of *D*-function of non-local field.

Adopting the notations used in Yukawa's paper, the field operators for scalar theory are expressed as follows:

$$U(X_\mu, r_\mu) = \int \cdots \int (dk)^4 (dl)^4 \bar{U}(k_\mu, l_\mu) \times \exp(ik_\mu X^\mu) \prod_\mu \delta(r_\mu + l_\mu),$$

$$U^*(X_\mu, r_\mu) = \int \cdots \int (dk')^4 (dl')^4 \bar{U}^*(k_\mu', l_\mu') \times \exp(ik_\mu' X^\mu) \prod_\mu \delta(r_\mu - l_\mu'), \quad (1)$$

where

$$\bar{U}(k_\mu, l_\mu) = U(k_\mu, l_\mu)\delta(k_\mu k^\mu + \varkappa^2) \times \delta(k_\mu l^\mu)\delta(l_\mu l^\mu - \lambda^2),$$

$$\bar{U}^*(k_\mu', l_\mu') = U^*(k_\mu', l_\mu')\delta(k_\mu' k^{\mu\prime} + \varkappa^2) \times \delta(k_\mu' l^{\mu\prime})\delta(l_\mu' l^{\mu\prime} - \lambda^2). \quad (2)$$

These latter operators satisfy the following commutation relation;

$$[U(k_\mu, l_\mu),\ U^*(k_\mu', l_\mu')]$$
$$= -\frac{k_4}{|k_4|}\delta(k_\mu k^\mu + \varkappa^2)\cdot\delta(k_\mu l^\mu)\times$$
$$\times\delta(l_\mu l^\mu - \lambda^2)\prod_\mu \delta(k_\mu - k_\mu')\delta(l_\mu - l_\mu'). \quad (3)$$

From Eq. (1), (2) and (3), the following relation is derived,

$$[U(X_\mu, r_\mu),\ U^*(X_\mu', r_\mu')] =$$
$$-\prod_\mu \delta(r_\mu + r_\mu')\cdot\delta(r_\mu r^\mu - \lambda^2)\times$$
$$\times\int (dk)^4 \frac{k_4}{|k_4|}\delta(k_\mu k^\mu + \varkappa^2)\delta(k_\mu r^\mu)\times$$
$$\times\exp\{ik_\mu(X^\mu - X^{\mu\prime})\}$$
$$\equiv \prod\delta(r_\mu + r_\mu')\delta(r_\mu r^\mu - \lambda^2)\cdot D'(X^\mu - X^{\mu\prime\prime}, r_\mu). \quad (4)$$

It is noticed, firstly, the new D-function depends only on *the mean position difference* $X^\mu - X^{\mu\prime}$ and reduces to the ordinary one

$$D(X^\mu - X^{\mu\prime}) = -\int (dk)^4 \frac{k^4}{|k_4|}\delta(k_\mu k^\mu + \varkappa^2)\times$$
$$\times\exp\{ik_\mu(X^\mu - X^{\mu\prime})\}, \quad (4')$$

when λ tends to zero. Secondly, it is zero unless *the internal coordinates* satisfy the relation $r_\mu = -r_\mu'$.

Integrating on the k_4 variable, $D'(X^\mu, r_\mu)$ reduces to

$$D'(X^\mu, r_\mu) = \frac{1}{2}\int dk \frac{1}{K}\delta(kr - Kr^4)\times$$
$$\times\{\exp(i(kX - KX^4)) - \exp(-i(kX - KX^4))\}$$

where

$$X_\mu = (X, X_4),\ r_\mu = (r, r_4),\ k_\mu = (k, k_4),$$
$$K = \sqrt{k^2 + \varkappa^2}.$$

As this function is relativistically invariant, we can adopt the Lorentz coordinate system, in which $r^4 = r_4 = 0$, $|r| = r = \lambda$. In addition we take the direction of r as z-axis. Then we obtain

$$D'(X_\mu, r_\mu) = \frac{1}{2\lambda}\int dk \frac{1}{K}\delta(k_z)\{\exp(i(kX - KX^4))$$
$$-\exp(i(kX - KX^4))\}$$
$$= \frac{1}{2\lambda}\int dk_x dk_y \frac{1}{K}\{\exp(i(kX - KX^4))$$
$$-\exp(-i(kX - KX^4))\}$$
$$= \frac{i}{\lambda}\int_0^\infty\int_0^{2\pi} \frac{k dk d\varphi}{K}\sin(kR\sin\theta\cos\varphi - KX^4)$$
$$(r \perp k) \quad (5)$$
$$= -\frac{2\pi i}{\lambda}\int_0^\infty \frac{k dk}{K} J_0(kR\sin\theta)\sin KX^4$$
$$= -\frac{2\pi i}{\lambda}\frac{1}{R\sin\theta}\frac{\partial}{\partial(R\sin\theta)}R\sin\theta\times$$
$$\times\int_0^\infty (J_1(kR\sin\theta)\sin KX^4/K)dk,$$

where

$$R = |X|,\ r\cdot X = rR\cos\theta,\ k\cdot X = kR\sin\theta\cos\varphi.$$

This corresponds to the ordinary expression

$$D(X_\mu) = -\frac{4\pi i}{R}\int_0^\infty \frac{k dk}{K}\sin kR\sin KX^4$$
$$= 4\pi i\frac{1}{R}\frac{\partial}{\partial R}\int_0^\infty \frac{\cos kR\sin KX^4}{K}dk. \quad (5')$$

Though the integral in (5) cannot be expressed by an analytic function of R and X^4, it is remarkable that it depends on the angle between r and X. Especially when $\theta = 0$, it no more depends on R. This situation appears more clearly, when we make the function corresponding to the δ-function of local field. Namely, in the coordinate system in which $r_4 = 0$, we obtain

$$\frac{\partial}{\partial X_4}D'(X_\mu, r_\mu)|_{X_4=0} = -\frac{i}{2\lambda}\int dk_x dk_y\times$$
$$\times\{\exp(ikX) + \exp(-ikX)\}$$
$$= ((2\pi)^2/i\lambda)\delta(X)\delta(Y), \quad (6)$$

corresponding to the ordinary one

$$\frac{\partial}{\partial X_4}D(X_\mu)|_{X_4=0} = ((2\pi)^3/i)\delta(X)\delta(Y)\delta(Z), \quad (6')$$

where

$$X = (X, Y, Z).$$

Eq. (5) or (6) becomes infinite when we make λ zero. This apparent difficulty can

be removed readily when combining D-function with $\delta(r_\mu r^\mu - \lambda^2)$ (see Eq. (4)) and noting $\delta(r_\mu r^\mu - \lambda^2)/\lambda = \delta(r-\lambda)/2\lambda^2$ tends to $\pi\delta(r)$ as $\lambda \to 0$. Eq. (6) does not coincide with Eq. (6′) as $\lambda \to 0$, although the not-yet-integrated form (4) does, showing pathological natures of D-functions. We summarize as follows; Field operators are commutable unless the internal coordinates are mutually antiparallel; if they are antiparallel, the commutators are independent of mean position difference, when the internal coordinates and mean position difference have the same direction; when they are perpendicular to each other, commutators are δ-function type as usual. Above results are hardly understandable from the standpoint that the internal coordinate has its physical meaning as the extension of a particle. But all these explorations are still pending to the future clarification.

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Effects of Nuclear Motion on the Fine and the Hyperfine Structure of Hydrogen

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The 16-component equation for a proton and an electron with Breit's interaction has been solved now by expansion in power series in α^2 successively ($\alpha = e^2/c\hbar$), not in $\beta(=m/M)$ as in my previous work reported[1], and for general values of the total angular momentum F. The treatment is essentially the same as that of Bechert and Meixner[2], but the process of approximation included there has become clearer in this work and some refinement in their results has been obtained.

The results have shown: The correction for the Rydberg energy and the fine structure of the hydrogen is given by the Bohr's simple reduced mass correction factor $(1+\beta)^{-1}$ as well as the Darwin's additional term which can be expressed, in units of mc^2, by $-(\beta\alpha^4/8n^4)(1+\beta)^{-3}$, in accord with every previous work[2),3),4)]. The result reported in my last letter for the energy of order $\beta^2\alpha^2$ in the same units was different from this, but it has been proved that some misunderstanding was involved in its reduction.

The correction factor for the hyperfine structure is given by $(1+\beta)^{-3}$ for the two extreme hyperfine levels for a given l: $j=l+1/2$, $F=l+1$ and $j=l-1/2$, $F=l-1$ as Breit and Meyerott[5)] and others[6)] have shown, but for the other two levels $j=l\pm 1/2$, $F=l$, a different factor $(1+\beta)^{-2}$ follows. The correction for the hyperfine separation of the ground state or more generally, of energy S state of the hydrogen atoms is not affected by this asymmetric factor and given by the factor $(1+\beta)^{-3}$ or $1-3\beta$ as usual, since the energy expression for the S state lacks a term which leads to this asymmetric factor.

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On the Nuclear Stars

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Recently, the nuclear evaporations with higher excitation energy have been discussed by many authors[1-6]. Thus it seems useful to compare with them.

First of all, HLP, in which various theories are checked in comparison with huge collection of experimental results, is the most important material in this field. In this paper it is concluded that as for the energy-temperature relation the Fermi-gas model is more preferable than to the liquid drop model. This conclusion is consistent with the empirical information on nuclear level densities[7]. Furthermore, if we assume that the smaller nuclear radius proposed by Serber et al.[8], the so-called Bardeen's remark[9),10] seems unnecessary (while HLP used the large nuclear radius of Bethe)[10].

HLP, HB, FY, CP and LC are all based on Weisskopf's evaporation theory[10),11]. On the contrary Goldberger[6] has adopted the Monte Carlo method throughout the reaction process induced by high energy neutrons and has tried to check the statistical evaporation theory. I think, however, that his check is somewhat questionable because he used the Monte Carlo method until regions of too low energy to accept its validity.

Next, CP, based on the liquid drop model, may be rejected by reason of its wrong energy distribution of star prongs, as discussed in HLP. Also the level density formula used in CP has little experimental support[7]. They have taken into account the fluctuation of numbers of evaporated particles, and have tried to interpret the 90Mev neutron induced reaction and negative meson stars. Nevertheless, if we include the effect of nuclear transparency[2),3),6),8),12], which gives very broad distribution of initial excitation energy of the nucleus, we can hardly obtain such results as given in CP. The differences between light and heavy nucleus-stars[2),13] can not be explained in CP. More refined treatments of meson stars are given by us[14] (as for light nucleus-stars, see ref.[15]).

HB and CP overlooked the change of binding energies in the course of evaporation process. This effect was included in FY and LC. FY has a defect that it is used rather crude approximations for various quantities. The treatment in LC is the most accurate and complete, though in this paper the fluctuation of numbers of emitted particles is not considered. The elementary considerations of this fluctuation are given in FY and ref. 16).

In highly excited nuclei, there exist some special circumstances (e.g., the thermal expansion[17], depression of Coulomb barrier height[18], change of nuclear surface tension[19], etc.), and these make the treatment of very large stars more complicated.

In short, the present stage of nuclear evaporation theory may be compared with that of the cascade theory in cosmic-ray physics, and it is most desirable that more accurate calculations which take account of all possible effect and use empirically well determined quantities (binding energies, etc.), should be performed.

In the star-problem the theory of light nucleus-stars is particularly poor. Heckrotte and Wolff[20] analyzed the carbon reactions on the basis of Weisskopf's theory. But the nuclei such as C, N or O are too light for the statistical theory to be valid, although the tendency of prong-number distribution[13] is explained from the statistical considerations[15].

Recently the striking differences between *n-p* and *p-p* scattering at high energies have been found[21]. Thus the calculations on nuclear transparency of high energy nucleons (refs. 2), 6), 8) and 12); these are based on old experimental results) should be revised. Since the *n-n* force is remained unknown at present, it is deeply desired to analyze experimentally and theoretically, e.g., *n-d* scattering at higher energy.

The author wish to thank Mr. Fujimoto for his helpful discussions.

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On the Positron Theory of Vacuum

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May 24, 1950

The problems of photon self-energy and gauge invariancy are the appropriate examples of the formal difficulties in the present quantum field theory and we have no satisfactory answer notwithstanding the various methods of remedy proposed by many authors, such as the mixed theory, the regulator method and other technical methods of calculations.

In these circumstances, we have a little opportunity to re-examine the positron theory of vacuum analysed by Dirac[1] and Heisenberg[2] et al. in some years ago and some reflections on this problem have been proposed by a few authors in our country.[3),4)]

We also treat this theory from the thoroughly covariant view point developed by Tomonaga and Schwinger and then clarify the situation of these problems.

At first, we consider the electron field under the influences of the external electromagnetic field satisfying the following Dirac's equation

$$[\gamma_\mu(\partial_\mu - ieA_\mu) + m]\psi(x) = 0, \tag{1}$$

and then construct the vacuum expectation value

$$<[\psi(x'), \bar{\psi}(x'')]>_0 = -S^{(1)}(x', x''), \tag{2}$$

by means of $S^{(1)}(x', x'')$ given by some integral equations. If there would exist no real electron field, this function is also given by the power series of e, that is

$$S^{(1)}(x', x'') = \sum_{n=0}^{\infty}(-ie)^n \int\cdots\int dx_1 \cdots\cdots dx_n \sum_{\substack{i=0 \\ x_0 = x'}}^{n} \bar{S}(x' - x_1)\gamma_\mu A_\mu(x_1) \cdots S^{(1)}(x_i - x_{i+1}) \cdots \gamma_\mu A_\mu(x_n)\bar{S}(x_n - x''). \tag{3}$$

Following Dirac's program, one can get the physically significant density matrix $K^{(1)}$ by means of subtraction of any singular term,

$$K^{(1)}(x', x'') = S^{(1)}(x', x'') - U^{(1)}(x', x''). \tag{4}$$

And we must determine $U^{(1)}$ by the requirements of the equation of motion and the conservation law. Although it will be more difficult to find this subtration terms

rigorously than in Heisenberg's first treatment, we can easily get the first several leading terms as follows:

$$U^{(1)}\left(x+\frac{z}{2}, x-\frac{z}{2}\right)=u\left(x+\frac{z}{2}, x-\frac{z}{2}\right)S^{(1)}(z)$$
$$+\frac{ie}{(2\pi)^3}\int_0^1 da\int (dp)e^{ipz}\Big[(4-7a)$$
$$\gamma_\mu\gamma_\nu(i\gamma_o p_o-m)F_{\mu\nu}(x-\frac{z}{2}+za)$$
$$+(5-8a)\gamma_\mu ip_\nu F_{\mu\nu}(x-\frac{z}{2}+za)$$
$$+2a(a-1)\gamma_\mu\frac{\partial F_{\mu\nu}(x-\frac{z}{2}+za)}{\partial x_\nu}\Big]$$
$$\delta'(p^2+m^2)u\left(x-\frac{z}{2}+za, x-\frac{z}{2}\right), \quad (5)$$

where $x'=x+\frac{z}{2}$, $x''=x-\frac{z}{2}$ and $u(x', x'')$ are given by the integral equation:

$$u(x', x'')=1+ie\int_{x''}^{x'}A_\lambda(x''')u(x''', x'')dx_\lambda'''. \quad (6)$$

Expanding $u(x', x'')$ by coupling constant e, we have

$$u\left(x+\frac{z}{2}, x-\frac{z}{2}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(ie)^n}{n!}\int_0^1 A_\alpha\left(x-\frac{z}{2}+zb_1^{\frac{1}{n}}\right)db_1\cdots$$
$$\int_0^1 A_\omega(x-\frac{z}{2}+zb_1^{\frac{1}{n}}b_2^{\frac{1}{n-1}}\cdots b_n)db_n z_\alpha\cdots z_\omega \quad (6')$$

and for the case of the product of this function by the field independent some functions, this is written approximately

$$u\left(x+\frac{z}{2}, x-\frac{z}{2}\right)$$
$$=\exp\left[ie\int_0^1 A_\lambda(x-\frac{z}{2}+xa)daz_\lambda\right]. \quad (7)$$

That is, the first term of (5) becomes

$$\exp\left[ie\int_0^1 A_\lambda da z_\lambda\right]S^{(1)}(z)$$
$$=\frac{1}{(2\pi)^3}\int (dp)e^{ipz}[i\gamma_\lambda(p_\lambda-e\int_0^1 A_\lambda du)-m]$$
$$\times\delta\left[(p-e\int_0^1 Adu)^2+m^2\right], \quad (8)$$

which coincides with the first term of Heisenberg's calculations and the subtraction terms of Umezawa and Kawabe's consideration.[4] However, for the other subtraction terms we can not modify so simply and the physical meaning of these terms are not so clear than the first.

According to this subtraction procedure (5), we have the gauge invariant results and no charge renormalization in the usual covariant theory. And the non-gauge terms, such as the photon self-energy, those of the γ-decay of neutral scalar, pseudo-scalar and -vector meson and the photon-photon scattering are given correctly in the subtraction terms as $S^{(1)}(x', x'')$, that is,

$$\frac{ie}{2}S_p[U^{(1)}(x,x)\gamma_\rho]A_\rho$$
$$=\frac{a}{2\pi}\int (dp)\left[\delta(p^2+m^2)+\frac{1}{2}p^2\delta'(p^2+m^2)\right]A_\rho^2$$
$$-\frac{2a}{3\pi}\left(\log\frac{P_0+P}{m}-1\right)J_\rho A_\rho,$$

$$\frac{f}{2}S_p[U^{(1)}(x, x)]_2\phi=\frac{a}{2\pi}fm\phi A_\lambda^2,$$

$$\frac{i}{2}\frac{f}{\mu}S_p[U^{(1)}(x, x)\gamma_5\gamma_\rho]_2\frac{\partial\phi}{\partial x_\rho}$$
$$=\frac{a}{3\pi}\frac{f}{\mu}\frac{\partial\phi}{\partial x_\rho}\sum_{cycl.}A_\lambda F_{\mu\nu},$$

and

$$\frac{ig}{2}S_p[U^{(1)}(x, x)\gamma_\rho]_3U_\rho=\frac{a}{12\pi}egA_\lambda^2A_\rho U_\rho. \quad (9)$$

Detailed account will appear in this journal.

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YOSHIO NISHINA

Dr. YOSHIO NISHINA, His Sixtieth Birthday

Dr. Yoshio Nishina's sixtieth birthday will come round on December 6th, 1950. On this happy occasion, I, as one of his old pupils, should like, by way of expressing my hearty congratulation, to celebrate not only what he has done in physics, but also the great and important part he has played and will still play in the development of general culture and science of this country.

When he graduated from Tokyo Imperial University in 1918, he intended to devote himself to the study of electric engineering. But after he entered the Institute of Physical and Chemical Research here, he came to be interested in pure physics, and so during his long stay in Europe from 1921 to 1928, too, he gave himself wholly up to that study.

Does it not show that he has a mind both very brilliant and flexible that Dr. Nishina who had been educated to be an electric engineer proved to be an excellent scholar in pure physics? His research works cover a very wide scope, extending over not only theoretical physics and experimental physics, but even biology, agriculture, and medical science. Even within the limit of physics, his research activities cover almost all its branches such as quantum theory, theory of elementary particles, X-rays, atomic nucleus, and cosmic rays,— the result of his extraordinary enthusiasm and energy to carry out research works.

It is needless as well as impossible to enumerate here one by one all the results of these many-sided researches of his, for he has done a very important and significant work which by itself will be enough to immortalize his name.

And that is his study of the scattering of hard X-rays or γ-rays by a free electron. During his stay in Europe, Dr. Nishina made a theoretical study of this problem with Dr. Klein, a Swedish theorist, and evolved from it the famous Klein-Nishina formula. How widely this formula is being used, how important a part it is now playing, in the field of nuclear and cosmic ray physics, is a well-known fact to anyone who has any knowledge of the present-day physics.

Apart from these excellent research achievements, his existence has a special significance to us physicists of Japan, firstly because it was no one but he that brought the seeds of new physics into this country and secondly because deep impressions and strong stimulations were given to young students and researchers by him through his character and his enthusiasm at once persistent and energetic. If it were not for him, the younger generation in every branch of science in Japan would have been quite at a loss.

Dr. Nishina, who had spent his days in Europe with Lord Rutherford at Cavendish Laboratory, and with Dr. Bohr at Copenhagen, had grown with the growth of new physics, absorbing a fresh and active atmosphere of its centres, and personally going through new methods of its study. So, when he came home later, he tried to introduce and transplant these methods and atmosphere into this country. How pleasant and exciting it was to us younger people to listen to his stories of his own experiences at those centres of new science in Europe! How deeply impressed and influenced we were by his zealous academic interest through his stories and deeds!

He did not only encourage younger people morally, but, as soon as Nishina Laboratory was made in 1931 in the Institute of Physical and Chemical Research after his return from Europe, he set about his great work of making this laboratory a research centre of atomic nucleus and cosmic rays. We can well imagine how hard it must have been to realize that

great plan in this country of those days, which was economically poor as well as lacking in general sympathy and understanding towards pure studies. A laboratory required for research in new physics, whether it be nuclear physics or cosmic ray physics, is a laboratory of a very large scale which is furnished with a monster of a crane above our head and a gigantic high-tension apparatus weighing as much as a few hundred tons, and which has an electric source and a machine shop attached to it besides—in short, it looks like a factory, so to speak. How laborious a task it must have been, and how painstakingly he must have worked, to enlighten the people and make them understand that a research in pure physics needed such a big "factory." But at last his earnest campaign of enlightenment succeeded, and a nuclear research laboratory was made in the Institute of Physical and Chemical Research and two cyclotrons were set there in 1935. The result of his campaign was not limited to that only, but some universities came to have their own "factories" of nuclear physics furnished with such majestic apparatuses as cyclotrons and so forth, for people at large were pursuaded by him and began at last to perceive the importance of the study of atomic nucleus.

Here in his Nuclear Research Laboratory, Dr. Nishina wanted to carry out his great plan of research, not only in pure nuclear physics, but also in its wide application to biology, to agriculture, and even to medical science. Here, however, the Pacific War broke out, and it became difficult for him to continue his research, much more so to carry out his new plans and new ideas. Thus his grand conception collapsed.

Since the end of the War he has been and is, as the president of the Scientific Research Institute that succeeded the former Physical and Chemical Research Institute, making efforts to carry out its aim by good management though under extraordinarily difficult conditions. Thus he has turned now his elastic and versatile ability and untiring energy from his academic life to a new life of an administrator of a research institute. And he is, I hope, competent enough to accomplish the difficult task which is, in our post-war Japan, even more important and difficult than research itself.

He is now also endeavouring, as vice-president of Japan Science Council, to contribute to science administration and to promote international co-operation in science. We now only wish and hope that the great scientist of world-wide fame who is also the father of new physics of Japan will be as healthy and as active as ever, because we expect much from him in these hard times of Japan and the world.

Sin-itiro TOMONAGA

Nov. 20th, 1950

Obituary Notice :

In the cource of the proof of this number, on Wednesday evening, January 10, 1951, Dr. Y. Nishina was stricken by a cancer of the liver. Death came a few months after going to hospital in Tokyo. Here we offer our cordial condolence.

Jan. 11, 1951 M. K.